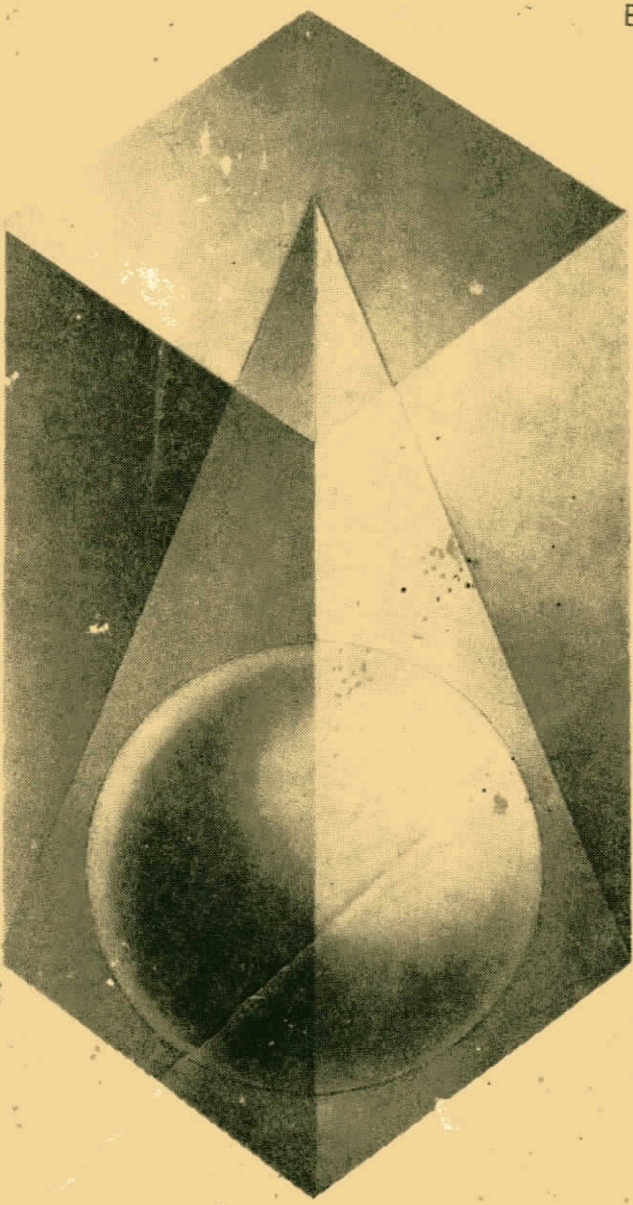
GEOMETRY

E.Z. SHUVALOVA



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E.Z. SHUVALOVA

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SYMBOLS USED IN THE BOOK

(AB)	The straight line passing through the points A and B.
$[AB]$	The line segment with the endpoints at A and B.
$\mid AB \mid$	The length of the segment $[AB]$.
$\rho\,(F_1, F_2)$	The distance between the figures F_1 and F_2.
$(\widehat{AB, CD})$	The angle between the lines (AB) and (CD).
$\angle$	Angle.
$\widehat{\quad}$	The magnitude of an angle.
$A \in F$ $(A \notin F)$	The point A belongs (does not belong) to the set F.
$F_1 \subset F_2$ $(F_1 \not\subset F_2)$	The set F_1 is contained (is not contained) in the set F_2.
$F_1 \cap F_2$	Intersection of the sets F_1 and F_2.
$F_1 \cup F_2$	Join of the sets F_1 and F_2.
$\cong$	Congruent; is congruent to.
$\backsim$	Is similar to.
$\parallel$	Is parallel to; parallel
$\perp$	Perpendicular; is perpendicular to.

$\forall$	For any.
$\exists$	Exists.
$\Rightarrow$	Follows.
$\Leftrightarrow$	Equivalent; is equivalent to
$\mathrm{pr}_{\alpha}F$	The projection of the figure F on the plane α.
$\vec{a}$, $\overrightarrow{AB}$	Vector.
$\lvert\vec{a}\rvert$, $\lvert\overrightarrow{AB}\rvert$	The length of vector.
$\widehat{(\vec{a}, \vec{b})}$	The angle between vectors.
$\vec{a}\cdot\vec{b}$, $\overrightarrow{AB}\cdot\overrightarrow{CD}$	Scalar product, or dot product, of the vectors $\vec{a}$ and $\vec{b}$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$.
$\vec{a}\times\vec{b}$, $[\vec{a}\vec{b}]$	Vector product, or cross product, of the vectors $\vec{a}$ and $\vec{b}$.
$\vec{a}\vec{b}\vec{c}$	Triple product of three vectors
(A, B, C)	The plane passing through the points A, B, C.
$M(x, y, z)$, $M = (x, y, z)$	Point with coordinates x, y, z in space.
$\vec{a}\,\{x, y, z\}$, $\vec{a} = \{x, y, z\}$	Vector with coordinates x, y, z.

CHAPTER 1

BASIC RELATIONS BETWEEN THE ELEMENTS OF AN ARBITRARY TRIANGLE. SOLVING OBLIQUE TRIANGLES

We shall consider here the basic cases of solving arbitrary triangles. The results obtained in this chapter we shall then use for solving problems in geometry and physics. To do this we shall need some trigonometric formulas which are derived below.

Sec. 1. The Law of Sines

Lemma. *The side of any triangle is equal to the product of the diameter of the circle described about the triangle* (or the circumcircle) *and the sine of the opposite angle.*

Given: $\triangle ABC$ and the radius R of the circumcircle.
Prove:

$$a=2R\sin\hat{A},\quad b=2R\sin\hat{B},\quad c=2R\sin\hat{C}. \tag{1}$$

Proof. 1. We shall first consider the case when the angle A is acute. Let us describe a circle about the triangle ABC (Fig. 1) and draw the diameter $[BD]$ from the vertex B (or C). We then join the point D to C and consider the triangle BCD. In this triangle $\widehat{BCD}=90^\circ$ as an angle subtended by the diameter and $\widehat{BDC}=\hat{A}$ as inscribed angles subtended by one and the same arc.

It follows from the triangle BCD that the length of the leg $[BC]$ is equal to the length of the hypotenuse $[BD]$, multiplied by the sine of the opposite angle, i.e. $a=2R\sin\hat{A}$.

The remaining two formulas for $\hat{B}<90^\circ$ and $\hat{C}<90^\circ$ are proved in a similar way.

2. Suppose the angle A is obtuse. We describe a circle about the triangle ABC (Fig. 2) and consider the triangle

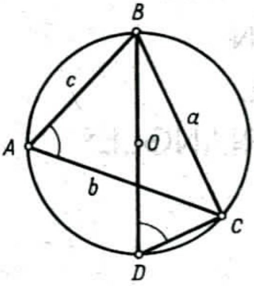


Fig. 1.

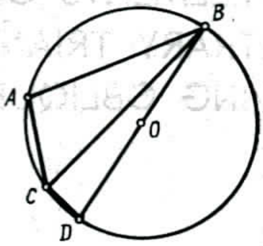


Fig. 2

BCD (whose side $[BD]$ is the diameter of the circumcircle) in which $\widehat{BCD} = 90°$. Consequently,

$$|BC| = |BD| \cdot \sin \widehat{BDC}.$$

Since $\sin \widehat{BDC} = \sin (180° - \widehat{BAC}) = \sin \widehat{BAC} = \sin \hat{A}$, we have $|a| = 2R \sin \hat{A}$.

3. If A is a right angle, then in the right-angled triangle ABC the side a is the hypotenuse and hence $|a| = 2R$, i.e. $|a| = 2R \sin 90°$. Thus, the lemma remains valid in this case as well.

Theorem (the law of sines). *The sides of any triangle are proportional to the sines of the opposite angles.*

Given: $\triangle ABC$.

Prove: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Proof. It follows from the lemma that

$$2R = \frac{a}{\sin A}, \quad 2R = \frac{b}{\sin B}, \quad 2R = \frac{c}{\sin C}$$

Hence,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{2}$$

Note. There is another way of deriving the formula (2) following from the equalities

$$S=\frac{1}{2}ab\sin\hat{C}=\frac{1}{2}bc\sin\hat{A}=\frac{1}{2}ac\sin\hat{B}.$$

As an application of the law of sines let us consider the proof of the theorem on the property of the bisector of the interior angle of a triangle.

Theorem. *The bisector of the interior angle of a triangle divides the opposite side into parts proportional to the adjacent sides.*

Given: $\triangle ABC$ and the bisector $[CD]$ of the angle C (Fig. 3).

Prove: $|AD|:|DB|=|AC|:|CB|$.

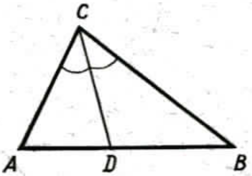


Fig. 3

Proof. The bisector $[CD]$ divides the angle C into two equal angles $\widehat{ACD}=\widehat{DCB}=\alpha$. Put $\widehat{ADC}=\beta$, then $\widehat{CDB}=180^\circ-\beta$. According to the law of sines we have in the triangles ACD and DBC

$$|AD|:|AC|=\sin\alpha:\sin\beta,$$
$$|DB|:|CB|=\sin\alpha:\sin\beta.$$

Consequently, $|AD|:|AC|=|DB|:|CB|$, which was required to be proved.

Sec. 2. The Law of Cosines

Theorem 1 (the law of cosines). *The square of the side of any triangle is equal to the sum of squares of two other sides less twice their product by the cosine of the angle between them.*

Given: $\triangle ABC$.

Prove: $a^2=b^2+c^2-2bc\cos\hat{A}$.

Proof. 1. Suppose the angle A is acute. Draw the altitude $[BD]$ in the triangle ABC, and consider the right-angled triangle BDC thus obtained (Fig. 4). In it

$$a^2=(b-b_1)^2+h^2. \qquad (1)$$

Let us now express the quantities b_1 and h in terms of the sides and angles of the triangle ABC. In the right-angled triangle ABD we have $h=c\sin\hat{A}$, $b_1=c\cos\hat{A}$.

Substituting them in the equality (1), we find that

$$a^2 = (b - c\cos\hat{A})^2 + c^2\sin^2\hat{A},$$

whence we get

$$a^2 = b^2 - 2bc\cos\hat{A} + c^2(\cos^2\hat{A} + \sin^2\hat{A}) = b^2 + c^2 - 2bc\cos\hat{A}.$$

2. Suppose the angle A is obtuse. From the vertex B let us drop the altitude $[BD]$ onto the extension of the side

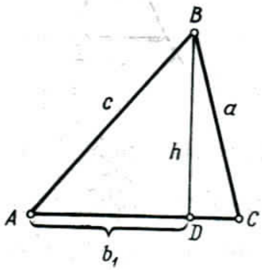


Fig. 4

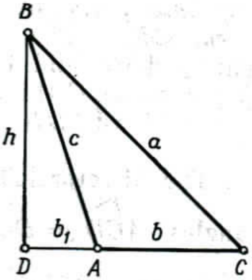


Fig. 5

$[AC]$ (Fig. 5). From the right-angled triangle BDC we have

$$a^2 = h^2 + (b + b_1)^2. \tag{2}$$

Expressing the quantities h and b_1 in terms of the sides and angles of the triangle ABC, we have

$$h = c\sin(180° - \hat{A}) = c\sin\hat{A},$$

$$b_1 = c\cos(180° - \hat{A}) = -c\cos\hat{A}.$$

Substituting these expressions in the equality (2) and carrying out all necessary simplifications, we get

$$a^2 = c^2\sin^2\hat{A} + (b - c\cos\hat{A})^2 = c^2(\sin^2\hat{A} + \cos^2\hat{A}) +$$

$$+ b^2 - 2bc\cos\hat{A} = b^2 + c^2 - 2bc\cos\hat{A}.$$

3. Suppose A is a right angle. In this case $\cos\hat{A} = 0$ and hence $b^2 + c^2 - 2bc\cos\hat{A} = b^2 + c^2$. Since according to the Pythagorean theorem $b^2 + c^2 = a^2$, we have

$$a^2 = b^2 + c^2 - 2bc\cos\hat{A}.$$

The law of cosines turns out to be a generalization of the Pythagorean theorem, since it yields the following corollaries.

Corollary 1. *The square of a side of a triangle is less than, equal to, or exceeds the sum of squares of two other of its sides depending on whether the opposite angle is acute, right, or obtuse.*

Proof. If $\hat{C} < 90°$, then $\cos \hat{C} > 0$ and

$$c^2 = a^2 + b^2 - 2ab \cos \hat{C} < a^2 + b^2.$$

If $\hat{C} = 90°$, then $\cos \hat{C} = 0$ and $c^2 = a^2 + b^2$.

If $\hat{C} > 90°$, then $\cos \hat{C} < 0$ and $c^2 = a^2 + b^2 - 2ab \times \times \cos \hat{C} > a^2 + b^2$.

The converse is also true.

Corollary 2. *The angle of a triangle is acute, right, or obtuse depending on whether the square of the opposite side of the triangle is, respectively, less than, equal to, or more than the sum of the squares of the two remaining sides.*

Proof. It follows from the law of cosines that

$$\cos \hat{C} = \frac{a^2 + b^2 - c^2}{2ab}.$$

If $c^2 < a^2 + b^2$, then $\cos \hat{C} > 0$ and hence $\hat{C} < 90°$. But if $c^2 = a^2 + b^2$, then $\cos \hat{C} = 0$, i.e. $\hat{C} = 90°$, and, finally, if $c^2 > a^2 + b^2$, then $\cos \hat{C} < 0$, i.e. $\hat{C} > 90°$.

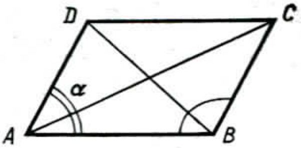


Fig. 6

Theorem 2. *In any parallelogram the sum of squares of the diagonals is equal to the sum of squares of all of its sides.*

Given: a parallelogram $ABCD$ and its diagonals $[AC]$ and $[BD]$ (Fig. 6).

Prove: $|AC|^2 + |BD|^2 = 2|AB|^2 + 2|BC|^2$.

Proof. Let $\widehat{DAB} = \alpha$, then $\widehat{ABC} = 180° - \alpha$.

According to the law of cosines, from $\triangle ADB$ and $\triangle ABC$ we have

$$|BD|^2 = |AD|^2 + |AB|^2 - 2|AB| \cdot |AD| \cdot \cos \alpha,$$

$$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB| \cdot |BC| \times \times \cos(180° - \alpha) = |AB|^2 + |BC|^2 + 2|AB| \times \times |AD| \cos \alpha,$$

since $|BC| = |AD|$. Adding these equations, we have

$$|AC|^2 + |BD|^2 = 2\,|AB|^2 + 2\,|BC|^2,$$

which was required to be proved.

Sec. 3. Expressing the Tangent of a Half-Angle in Terms of the Sides of a Triangle and Radius of the Inscribed Circle (or the Incircle)

Theorem. *The tangent of the half-angle of a triangle is equal to the quotient obtained in dividing the radius of the incircle by the difference between the semiperimeter and the opposite side.*

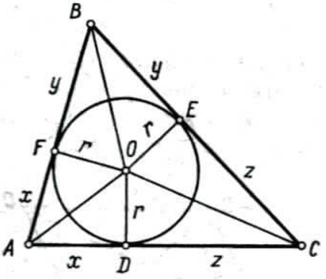


Fig. 7

Given: $\triangle ABC$; the radius r of the incircle, and the semiperimeter p.

Prove: $\tan\frac{\hat{A}}{2} = \frac{r}{p-a}$, $\tan\frac{\hat{B}}{2} = \frac{r}{p-b}$, $\tan\frac{\hat{C}}{2} = \frac{r}{p-c}$.

Proof. We draw the bisectors of the interior angles in the triangle ABC (Fig. 7). From the incentre O (i.e. the centre of the inscribed circle) we then drop perpendiculars $[OD]$, $[OE]$, and $[OF]$ on the sides of the triangle, the length of each perpendicular being equal to r. From the triangles thus obtained $AOD \cong AOF$, $COD \cong COE$, $BOF \cong BOE$ we have:

$$\tan\frac{\hat{A}}{2} = \frac{r}{|AD|}, \quad \tan\frac{\hat{B}}{2} = \frac{r}{|FB|}, \quad \tan\frac{\hat{C}}{2} = \frac{r}{|DC|}. \qquad (1)$$

Let us express $|AD|$, $|FB|$, $|DC|$ in terms of the sides of the $\triangle ABC$. Denoting $|AD| = |AF| = x$, $|FB| = |BE| = y$, $|DC| = |EC| = z$, we may write

$$a = z + y, \quad b = z + x, \quad c = x + y. \qquad (2)$$

Adding these equations, we get $a + b + c = 2\,(x + y + z)$, or

$$x + y + z = \frac{1}{2}\,(a + b + c) = p.$$

Subtracting each of the equations (2) from the last one, we find

$$p - a = x = |AD|, \quad p - b = y = |BF|,$$
$$p - c = z = |DC|,$$

and may rewrite the formulas (1) in the form

$$\tan\frac{\hat{A}}{2} = \frac{r}{p-a}, \quad \tan\frac{\hat{B}}{2} = \frac{r}{p-b}, \quad \tan\frac{\hat{C}}{2} = \frac{r}{p-c}, \qquad (3)$$

which was required to be proved.

The formulas (3) can be reduced to a more convenient form by expressing r in terms of the sides of the triangle ABC. To do this we shall make use of Hero's formula $S = \sqrt{p(p-a)(p-b)(p-c)}$ for the area of a triangle and of the formula $S = pr$ expressing the same area. Eliminating S from these equations, we get

$$r = \frac{\sqrt{p(p-a)(p-b)(p-c)}}{p} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}. \qquad (4)$$

Now from (3) and (4) it follows that

$$\tan\frac{\hat{A}}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}},$$
$$\tan\frac{\hat{B}}{2} = \sqrt{\frac{(p-a)(p-c)}{p(p-b)}},$$
$$\tan\frac{\hat{C}}{2} = \sqrt{\frac{(p-a)(p-b)}{p(p-c)}}. \qquad (5)$$

Sec. 4. Solving Oblique Triangles

To solve a triangle means to find the remaining angles and sides when sufficient of these have been given*. As is known, a triangle is specified by three basic elements, the side being at least one of them. In solving an oblique triangle the following four cases are possible:

(1) given three sides;
(2) given two sides and the angle between them;
(3) given two sides and the angle lying opposite them;
(4) given a side and two adjacent angles.

Let us consider each of them. As in any computational problem the correctness of the calculations performed is recommended to be checked by a formula not used in the solution of the problem. We shall call it a *checking formula.*

---

* Here and elsewhere "the side *a*" means the length of the side *a*.

Problem 1. Given three sides a, b, and c of a triangle. Find its angles.

Solution. The first method. Using the law of cosines $a^2 = b^2 + c^2 - 2bc \cos \hat{A}$, we find the angle $\hat{A}$:

$$\cos \hat{A} = \frac{b^2 + c^2 - a^2}{2bc}.$$

Using the law of sines $a : \sin \hat{A} = b : \sin \hat{B}$, we find the second angle $\hat{B}$:

$$\sin \hat{B} = \frac{b \sin \hat{A}}{a}.$$

The third angle $\hat{C}$ is found as a supplement to the angles $\hat{A}$ and $\hat{B}$:

$$\hat{C} = 180° - (\hat{A} + \hat{B}).$$

The checking formula: $a : \sin \hat{A} = c : \sin \hat{C}$.

The second method. Find the semiperimeter of the triangle $p = \frac{1}{2}(a + b + c)$ and the differences $p - a$, $p - b$, $p - c$. Using the formulas (5), we compute the required angles:

$$\tan \frac{\hat{A}}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}},$$

$$\tan \frac{\hat{B}}{2} = \sqrt{\frac{(p-a)(p-c)}{p(p-b)}},$$

$$\tan \frac{\hat{C}}{2} = \sqrt{\frac{(p-a)(p-b)}{p(p-c)}}.$$

The checking formula: $\hat{A} + \hat{B} + \hat{C} = 180°$.

The problem has a solution, which is a unique one, only if the sum of any two sides is more than the third side, i.e.

$$a + b > c, \quad b + c > a, \quad a + c > b.$$

If at least one of these conditions is not satisfied, then there is no solution at all.

Problem 2. Given two sides a and b $(a > b)$, and the angle $\hat{C}$. Find the side c and the angles $\hat{A}$ and $\hat{B}$.

Solution. Using the law of cosines, we find the side c:

$$c = \sqrt{a^2 + b^2 - 2ab \cos \hat{C}}.$$

The angles are found by the law of sines. It follows from the condition $a > b$ that the angle $\hat{B}$ is acute. Therefore, we first find the angle $\hat{B}$ by the formula

$$\sin \hat{B} = \frac{b \sin \hat{C}}{c}$$

and then the angle $\hat{A}$ as a supplementary angle to the angles $\hat{B}$ and $\hat{C}$

$$\hat{A} = 180° - (\hat{B} + \hat{C}).$$

The checking formula: $a : \sin \hat{A} = b : \sin \hat{B}$.

The problem has a solution, which is unique, if $\hat{C} < 180°$.

Note. The angle $\hat{A}$ may be of any magnitude if $\hat{C} < 90°$. Therefore, if prior to determining the angle $\hat{B}$ the angle $\hat{A}$ were found by the formula

$$\sin \hat{A} = \frac{a \sin \hat{C}}{c},$$

we would be unable to uniquely determine the magnitude of the angle $\hat{A}$ by the known value of its sine without additional investigations.

Problem 3. Given the side a and the angles $\hat{B}$ and $\hat{C}$. Find the sides b and c, and the angle $\hat{A}$.

Solution. We first determine the angle $\hat{A}$:

$$\hat{A} = 180° - (\hat{B} + \hat{C}).$$

Using the law of sines twice, we find b and c:

$$b = \frac{a \sin \hat{B}}{\sin \hat{A}}, \quad c = \frac{a \sin \hat{C}}{\sin \hat{A}}.$$

The checking formula: $a^2 = b^2 + c^2 - 2bc \cos \hat{A}$.

The problem has a solution, which is unique, if $\hat{B} + \hat{C} < 180°$.

Problem 4. Given two sides a and b and the angle $\hat{A}$. Find the side c and the angles $\hat{B}$ and $\hat{C}$.

Solution. Using the law of sines $\sin \hat{B} = \frac{b \sin \hat{A}}{a}$, we compute the angle $\hat{B}$ and then the angle $\hat{C}$ is found as a supplementary angle to $\hat{A}$ and $\hat{B}$:

$$\hat{C} = 180° - (\hat{A} + \hat{B}).$$

Using the law of sines $c = \frac{a \sin \hat{C}}{\sin \hat{A}}$, once again, we find the side c.

The checking formula: $a^2 = b^2 + c^2 - 2bc \cos \hat{A}$.

Let us notice how the angle $\hat{B}$ is calculated using the found value of $\sin \hat{B}$. Since two angles correspond to one and the same value of sine, there arises a question whether both of these angles are suitable or only one of them and which one.

We shall give an answer to this question by comparing the given sides a and b.

1. Suppose $a > b$. Then $a > b \sin \hat{A}$, whence $\sin \hat{B} = \frac{b \sin \hat{A}}{a} < 1$. The problem has a unique solution, since the angle $\hat{B}$ may be only acute independent of whether the given angle $\hat{A}$ is acute, right, or obtuse.

2. Suppose $a < b$. Then $\hat{A} < \hat{B}$. The problem has a solution provided $\sin \hat{B} = \frac{b \sin \hat{A}}{a} \leqslant 1$, or $b \sin \hat{A} \leqslant a$. If $b \sin \hat{A} = a$, then $\sin \hat{B} = 1$, which means that the required triangle is a right-angled one. If now $b \sin \hat{A} < a$, the problem has two solutions, since in this case two values should be taken for the angle $\hat{B}$, which will give us two triangles, one of them being acute-angled, the other obtuse.

If, finally, $b \sin \hat{A} > a$, then $\sin \hat{B} > 1$, and the problem has no solution.

3. Suppose $a = b$. Then $\hat{A} < 90^\circ$ and $\hat{B} < 90^\circ$. The problem has a unique solution.

Sec. 5. Measuring Distances Between "Inaccessible" Objects

It is not always possible to directly measure the distance between two objects A and B. For instance, it is impossible to do this when the observer is situated on one bank of a river and one or both objects on the other, and there is no bridge across the river. We also cannot measure directly the height of some vertical objects such as a hill or a post located across the river.

A point A is said to be accessible (or approachable) or inaccessible depending on whether or not the observer can approach it.

Thus, the distance between two points A and B can be measured directly if and only if these points are approachable. In carrying out such measurements we usually neglect the curvature of Earth's surface, since the areas in question are too small, and we consider them to be flat.

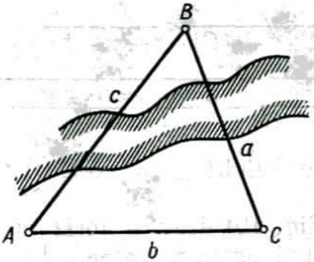


Fig. 8

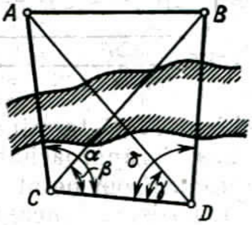


Fig. 9

Let us solve several simple problems on computing distances and heights.

Problem 1. Determine the distance from an approachable point A to an inaccessible point B.

Solution. Let us select an approachable point C and, by fixing it, measure directly the length of the line segment $[AC]$, which is called the *basis* (Fig. 8).

Let $|AC| = b$. Using a theodolite or an astrolabe, we determine the angles $\widehat{BAC}$ and $\widehat{BCA}$. In the triangle ABC the side $[AC] = b$ and the angles $\hat{A}$ and $\hat{C}$ are known (see Problem 3 of Sec. 4), hence

$$|AB| = c = \frac{b \sin \hat{C}}{\sin (\hat{A} + \hat{C})}.$$

Problem 2. Determine the distance between two inaccessible objects A and B.

Solution. Take two approachable points C and D, from which the points A and B are seen. Then we measure directly the distance $|CD| = a$ and the angles $\widehat{ACD} = \alpha$, $\widehat{ADC} = \gamma$,

$\widehat{BCD} = \beta$, and $\widehat{CDB} = \delta$ (Fig. 9) and compute the distances $|AC|$ and $|BC|$ (see Problem 3 of Sec. 4)

$$|AC| = \frac{a \sin \gamma}{\sin(\gamma + \alpha)}, \quad |BC| = \frac{a \sin \delta}{\sin(\beta + \delta)}.$$

Now, in the triangle ABC the sides $[AC]$ and $[BC]$, and the angle between them are known. By the law of cosines

$$|AB| = \sqrt{|AC|^2 + |CB|^2 - 2|CB| \cdot |AC| \cdot \cos(\alpha - \beta)} =$$

$$= a \sqrt{\frac{\sin^2 \gamma}{\sin^2(\alpha + \gamma)} + \frac{\sin^2 \delta}{\sin^2(\beta + \delta)} - 2\frac{\sin \gamma \sin \delta \cos(\alpha - \beta)}{\sin(\alpha + \gamma) \sin(\beta + \delta)}}.$$

Problem 3. Determine the height of a vertical object whose base is approachable.

Solution. From point A (Fig. 10) draw a horizontal line $[AC]$ (the basis) and measure its length. Suppose $|AC| = b$.

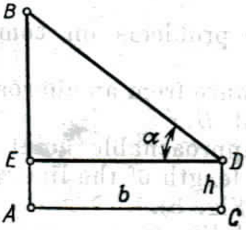


Fig. 10

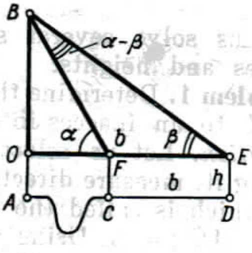


Fig. 11

Place then an angle measuring device at point C and measure $\widehat{BDE}$. Let $\widehat{BDE} = \alpha$, $|CD| = h$, where h is the height of the device. In the right-angled triangle BED the side $[BE]$ is a leg, therefore $|BE| = b \tan \alpha$ and $|AB| = b \tan \alpha + h$.

Problem 4. Determine the height of a vertical object whose base is inaccessible.

Solution. Take the basis $[CD]$ on a horizontal locality (Fig. 11) and measure its length. Let $|CD| = b$. We then place an angle measuring device in succession at points C

and D to measure the angles of inclination of the straight lines (BF) and (BE), respectively, to the horizontal plane

Let $\widehat{BFO} = \alpha$, $\widehat{BEO} = \beta$, $|DE| = |CF| = h$, where h is the height of the angle measuring device. In the triangle BFE the side $|FE| = b$, and the angles $\widehat{BFE} = 180° - \alpha$ and $\widehat{BEF} = \beta$ (see Problem 3 of Sec. 4). Therefore,

$$|BF| = \frac{b \sin \beta}{\sin (\alpha - \beta)}.$$

From the right-angled triangle BOF we now get

$$|OB| = \frac{b \sin \beta \sin \alpha}{\sin (\alpha - \beta)} \text{ and } |AB| = |OB| + h = \frac{b \sin \beta \sin \alpha}{\sin (\alpha - \beta)} + h.$$

Sec. 6. Other Problems on Solving Oblique Triangles

As is known, the triangle may be specified not only by its basic elements. We can construct a triangle given a median and two sides, or three medians, or a side, height (or altitude) and an angle, etc.

Let us consider a few problems on solving triangles specified by non-basic elements.

Problem 1. Given the triangle ABC with the angles $\hat{A}$ and $\hat{B}$, and the perimeter $2p = a + b + c$. Find the sides a, b, c, and the angle $\hat{C}$.

Solution. We first find the angle $\hat{C}$ and then, using the lemma formulated in Sec. 1, write the following relations:

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} = 2R.$$

Whence, using the property of equal relations, we find the radius of the circumcircle:

$$2R = \frac{a + b + c}{\sin \hat{A} + \sin \hat{B} + \sin \hat{C}},$$

i.e.

$$R = \frac{p}{\sin \hat{A} + \sin \hat{B} + \sin \hat{C}} = \frac{p}{4 \sin (\hat{A} + \hat{B})/2 \cdot \cos (\hat{A}/2) \cdot \cos (\hat{B}/2)}.$$

Using the law of sines, we now compute

$$a = 2R\sin\hat{A} = \frac{p\sin\hat{A}}{2\sin(\hat{A}+\hat{B})/2\cdot\cos(\hat{A}/2)\cdot\cos(\hat{B}/2)},$$

$$b = 2R\sin\hat{B} = \frac{p\sin\hat{B}}{2\sin(\hat{A}+\hat{B})/2\cdot\cos(\hat{A}/2)\cdot\cos(\hat{B}/2)},$$

$$c = 2R\sin\hat{C} = \frac{p\sin\hat{C}}{2\sin(\hat{A}+\hat{B})/2\cdot\cos(\hat{A}/2)\cdot\cos(\hat{B}/2)}.$$

It is easy to see that this solution is unique.

Problem 2. Given the triangle ABC with the angles $\hat{A}$ and $\hat{B}$ and the product of the sides $a\cdot b = n$. Find the sides a, b, c, and the angle $\hat{C}$.

Solution. We find $\hat{C} = 180° - (\hat{A} + \hat{B})$. Since $\frac{a}{b} = \frac{\sin\hat{A}}{\sin\hat{B}}$, we have the following system of two equations

$$\begin{cases} ab = n, \\ \dfrac{a}{b} = \dfrac{\sin\hat{A}}{\sin\hat{B}}, \end{cases}$$

from which we determine a and b:

$$a = \sqrt{n\sin\hat{A} : \sin\hat{B}}, \quad b = \sqrt{n\sin\hat{B} : \sin\hat{A}}.$$

We now find the side c:

$$c = \sin\hat{C}\cdot\sqrt{n : \sin\hat{B}\cdot\sin\hat{A}}.$$

The problem has a unique solution.

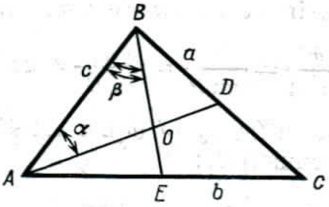


Fig. 12

Problem 3. Given the triangle ABC with the medians $|AD| = m_a$, $|BE| = m_b$, and the side $|AB| = c$. Find the sides a, b and the angles $\hat{A}$, $\hat{B}$, and $\hat{C}$ (Fig. 12).

Solution. The three sides of the triangle AOB (where O is the point of intersection of the medians) are known: $|AO| = \frac{2}{3} m_a$, $|OB| = \frac{2}{3} m_b$ and $|AB| = c$.

Solving this triangle (see Problem 1 of Sec. 4), we find

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(p-c)\left(p-\frac{2}{3}m_a\right)}{p\left(p-\frac{2}{3}m_b\right)}},$$

$$\tan\frac{\beta}{2} = \sqrt{\frac{(p-c)\left(p-\frac{2}{3}m_b\right)}{p\left(p-\frac{2}{3}m_a\right)}},$$

where $\alpha = \widehat{BAO}$, $\beta = \widehat{ABO}$,

$$p = \frac{1}{2}\left(c + \frac{2m_a}{3} + \frac{2m_b}{3}\right),$$

$$p - c = \frac{1}{2}\left(\frac{2m_a}{3} + \frac{2m_b}{3} - c\right),$$

$$p - \frac{2}{3}m_a = \frac{1}{2}\left(c - \frac{2m_a}{3} + \frac{2m_b}{3}\right),$$

$$p - \frac{2}{3}m_b = \frac{1}{2}\left(c + \frac{2m_a}{3} - \frac{2m_b}{3}\right).$$

Now, in each of the triangles ABD and ABE we know two sides and the angle between them (see Problem 2 of Sec. 4). Solving the triangle ABD ($|AB| = c$, $|AD| = m_a$, $\widehat{DAB} = \alpha$), we find

$$|BD| = \frac{a}{2} = \sqrt{c^2 + \frac{4}{9}m_a^2 - \frac{4}{3}cm_a\cos\alpha}$$

and

$$a = \frac{2}{3}\sqrt{9c^2 + 4m_a^2 - 12cm_a\cos\alpha},$$

$$\sin\widehat{ABD} = \frac{2m_a\sin\alpha}{a}.$$

Analogously, solving the triangle ABE ($|AB| = c$, $|BE| = m_b$, $\widehat{ABE} = \beta$), we find

$$|AE| = \frac{b}{2} = \sqrt{c^2 + \frac{4}{9}m_b^2 - \frac{4}{3}cm_b\cos\beta}$$

and

$$b=\frac{2}{3}\sqrt{9c^2+4m_b^2-12cm_b\cos\beta}\ ,\quad \sin\widehat{BAE}=\frac{2m_b\sin\beta}{b}.$$

Thus we have found the sides a and b, and the angles $\hat{A}$ and $\hat{B}$. The angle $\hat{C}=180^\circ-(\hat{A}+\hat{B})$.

The checking formula:

$$\tan\frac{\hat{C}}{2}=\sqrt{\frac{(p-a)(p-b)}{p(p-c)}}.$$

The above method of solution is not unique. The following method may be used as well.

In the triangle ABC we draw a third median $[CF]$ and on its extension lay off the line segment $[FK]\cong[FO]$ (Fig. 13).

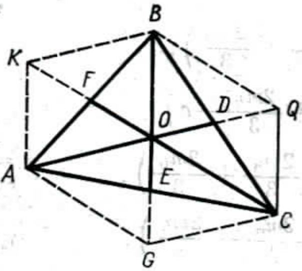


Fig. 13

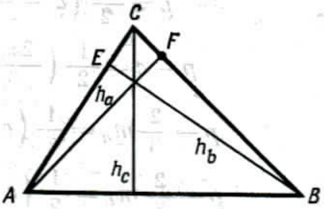


Fig. 14

The quadrilateral $AOBK$ is a parallelogram, since its diagonals are bisected by the point of their intersection. In this parallelogram the following elements are known: the sides $|AO|=\frac{2}{3}m_a$, $|OB|=\frac{2}{3}m_b$ and the diagonal $|AB|=c$.

Using Theorem 2 of Sec. 2, we find the second diagonal:

$$|OK|=|OC|=\frac{2}{3}m_c=$$

$$=\sqrt{2|OA|^2+2|OB|^2-|AB|^2}=$$

$$=\frac{1}{3}\sqrt{8m_a^2+8m_b^2-9c^2}.$$

On the extensions of the medians $[AD]$ and $[BE]$ we now lay off line segments $[DQ]\cong[OD]$ and $[EG]\cong[EO]$, respectively. The quadrilaterals $BOCQ$ and $AOCG$ are parallelograms, in each of which the sides and one of the diagonals are known. Using Theorem 2 of Sec. 2 once again we find

$$|BC| = a = \sqrt{2|OC|^2 + 2|OB|^2 - |OA|^2} = \\ = \frac{1}{3}\sqrt{8m_c^2 + 8m_b^2 - 4m_a^2},$$

$$|AC| = b = \sqrt{2|OC|^2 + 2|OA|^2 - |OB|^2} = \\ = \frac{1}{3}\sqrt{8m_c^2 + 8m_a^2 - 4m_b^2}.$$

Thus, we know the three sides of the triangle ABC. Solving Problem 1 of Sec. 4, we find the angles $\hat{A}$, $\hat{B}$, and $\hat{C}$.

Problem 4. Given the triangle ABC with the heights h_a, h_b, h_c. Find the sides and the angles of the triangle (Fig. 14).

Solution. From the formulas for the area of a triangle

$$S = \frac{1}{2}ah_a, \quad S = \frac{1}{2}bh_b, \quad S = \frac{1}{2}ch_c$$

it follows that

$$a : \frac{1}{h_a} = b : \frac{1}{h_b} = c : \frac{1}{h_c} = 2S.$$

It is clear from these relations, that the triangle ABC is similar to the triangle $A_1B_1C_1$ with the sides $a_1 = 1/h_a$, $b_1 = 1/h_b$, $c_1 = 1/h_c$; hence, $\hat{A} = \hat{A}_1$, $\hat{B} = \hat{B}_1$, $\hat{C} = \hat{C}_1$.

Solving the triangle $A_1B_1C_1$ (Problem 1 of Sec. 4), we find

$$\tan\frac{\hat{A}_1}{2} = \sqrt{\frac{(p-b_1)(p-c_1)}{p(p-a_1)}},$$

$$\tan\frac{\hat{B}_1}{2} = \sqrt{\frac{(p-a_1)(p-c_1)}{p(p-b_1)}},$$

$$\tan\frac{\hat{C}_1}{2} = \sqrt{\frac{(p-a_1)(p-b_1)}{p(p-c_1)}},$$

where $p = \frac{1}{2}(a_1 + b_1 + c_1)$.

Considering in succession the right-angled triangles BCE ($|BE| = h_b$), ACF ($|AF| = h_a$) and ABF ($|AF| = h_a$), we find:

$$a = |BC| = \frac{h_b}{\sin\hat{C}}, \quad b = |AC| = \frac{h_a}{\sin\hat{C}}, \quad c = |AB| = \frac{h_a}{\sin\hat{B}}.$$

Sec. 7. Worked Problems

Many problems in plane geometry, physics and mechanics are reduced to solving triangles for which purpose a figure is divided into a number of triangles so that the required element is incorporated in one of them (desirably, as the basic one).

If in such a triangle sufficient number of elements are given, then, by solving this triangle, we shall find the required element. If this number is not sufficient, then, by solving in succession a "chain" of triangles, we first find the missing elements and then, use them to find the required element.

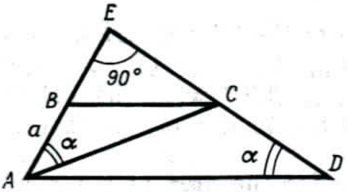


Fig. 15

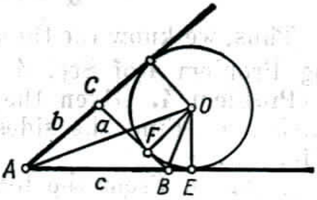


Fig. 16

Let us illustrate this by the following examples.

Example 1. The non-parallel sides of a trapezium are perpendicular to each other. One of them, which is equal to a, forms an angle α ($\alpha < 45^\circ$) with a diagonal, the other being inclined to the lower base at the same angle (Fig. 15). Find the median line of the trapezium.

Solution. By hypothesis, $\widehat{AED} = 90^\circ$, $\widehat{BAC} = \widehat{ADC} = \alpha$ and $|AB| = a$. The median line of the trapezium $|MN| = (|AD| + |BC|)/2$; $[BC]$ and $[AD]$ are the sides of the triangles ABC and ACD.

In the triangle ABC sufficient number of elements are known:

$$|AB| = a, \quad \widehat{BAC} = \alpha, \quad \widehat{ACB} = 90^\circ - 2\alpha, \quad \widehat{ABC} = 90^\circ + \alpha.$$

In the triangle ACD only the angles are known:

$$\widehat{CAD} = 90^\circ - 2\alpha, \quad \widehat{ADC} = \alpha, \quad \widehat{ACD} = 90^\circ + \alpha,$$

i.e. the number of elements is not sufficient. Therefore, prior to solving this triangle, we have to find one of the sides. Naturally, this side is $[AC]$ which will be determined when solving the triangle ABC.

Thus, solving the triangle ABC (Problem 3 of Sec. 4), we have

$$|BC| = \frac{|AB|\cdot \sin\alpha}{\sin(90^\circ - 2\alpha)} = \frac{a\sin\alpha}{\cos 2\alpha},$$

$$|AC| = \frac{|AB|\cdot\sin(90^\circ+\alpha)}{\sin(90^\circ-2\alpha)} = \frac{a\cos\alpha}{\cos 2\alpha}.$$

From the triangle ACD we now get

$$|AD| = \frac{|AC|\cdot\sin(90^\circ+\alpha)}{\sin\alpha} = \frac{a\cos^2\alpha}{\sin\alpha\cos 2\alpha}.$$

Hence,

$$|MN| = \frac{1}{2}(|AD| + |BC|) = \frac{a}{2\cos 2\alpha\sin\alpha}.$$

Example 2. In the triangle ABC with the side $|AB| = c$ and the angles $\hat{A} = \alpha$ and $\hat{B} = \beta$ a circle is inscribed externally which touches the side $[BC]$. Determine the radius of the circle.

Solution. The centre O of the circle lies in the point of intersection of the bisectors of the angles BAC and CBE (Fig. 16). Consequently, $\widehat{BAO} = \widehat{OAC} = \frac{\alpha}{2}$, $\widehat{FOB} = \widehat{BOE} = \frac{\beta}{2}$. Let us now draw $[OE]$ perpendicular to $[AE]$ and $[OF]$ perpendicular to $[CB]$. It is obvious, that $R = |OE| = |OF|$.

Thus, the required radius is the side of the right-angled triangle OBE ($\triangle OBE \cong \triangle OBF$), in which only one element is known, i.e. $\widehat{BOE} = \frac{\beta}{2}$, which is insufficient for solving the triangle. Therefore, let us first find its side. As is clear from Fig. 16, $[OB]$ is one of the sides of the triangle AOB, in which $AB = c$, $\widehat{OAB} = \alpha/2$, and $\widehat{AOB} = 90^\circ - \alpha/2 - \beta/2$.

Solving the triangle AOB, we find $|OB|$:

$$|OB| = \frac{|AB|\cdot\sin\widehat{OAB}}{\sin\widehat{AOB}} = \frac{c\sin\dfrac{\alpha}{2}}{\sin\left(90^\circ-\dfrac{\alpha}{2}-\dfrac{\beta}{2}\right)} = \frac{c\sin\dfrac{\alpha}{2}}{\cos\left(\dfrac{\alpha}{2}+\dfrac{\beta}{2}\right)},$$

and then from the triangle OBE we get

$$R = |OE| = |OB|\cdot\cos\frac{\beta}{2} = \frac{c\sin\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}}{\cos\left(\dfrac{\alpha}{2}+\dfrac{\beta}{2}\right)}.$$

It follows from the results obtained that there exists a solution for any triangle, since $0^\circ < \alpha + \beta < 180^\circ$ and $0^\circ < (\alpha + \beta)/2 < 90^\circ$.

Example 3. A body is acted upon by the forces $\vec{F}_1$ and $\vec{F}_2$. Find the resultant force $\vec{F}$ and the angles between $\vec{F}$ and $\vec{F}_1$, $\vec{F}$ and $\vec{F}_2$ if $|\vec{F}_1| = a$ N, $|\vec{F}_2| = b$ N, and $(\widehat{\vec{F}_1, \vec{F}_2}) = \alpha < 180^\circ$.

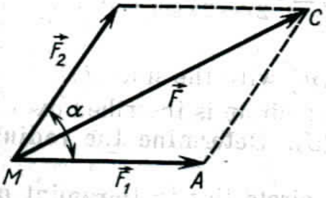


Fig. 17

Solution. As is known, the resultant force is directed along the diagonal of a parallelogram constructed on the vectors $\vec{F}_1$ and $\vec{F}_2$ and is numerically equal to the length of this diagonal. Hence, the problem is reduced to solving the triangle MAC (Fig. 17), in which the following elements are known: the sides $|MA| = a$, $|AC| = b$, and $\widehat{MAC} = 180^\circ - \alpha$ (Problem 2 of Sec. 4). Solving this triangle, we find:

$$|\vec{F}| = |MC| = \sqrt{a^2+b^2-2ab\cos(180^\circ-\alpha)} = \sqrt{a^2+b^2+2ab\cos\alpha},$$

$$\sin\widehat{AMC} = \frac{b\sin\alpha}{\sqrt{a^2+b^2+2ab\cos\alpha}}, \quad \widehat{BMC} = \alpha - \widehat{AMC},$$

where $\widehat{AMC} < 90^\circ$, since $a > b$.

Note. Let us check that $\sin \widehat{AMC} \leqslant 1$. From the inequality

$$\frac{b \sin \alpha}{\sqrt{a^2 + b^2 + 2ab \cos \alpha}} \leqslant 1$$

we get

$$b^2 \sin^2 \alpha \leqslant a^2 + b^2 + 2ab \cos \alpha, \quad \text{or}$$

$$b^2 \cos^2 \alpha + a^2 + 2ab \cos \alpha \geqslant 0.$$

The latter is obvious, since $b^2 \cos^2 \alpha + a^2 + 2ab \cos \alpha = (b \cos \alpha + a)^2$.

Example 4. A light beam pierces a plane-parallel plate. Determine the shift of this beam due to its passage through the plate if the angle of incidence is equal to α and the thickness of the plate is equal to d.

Solution. On meeting the plate at point A the beam $[MA)$ will change its direction and move along a straight line (AB) at an angle β to the perpendicular $[AC]$ (Fig. 18). Leaving the plate at point B the beam will propagate along the straight line (BN) inclined to (AC) at an angle γ.

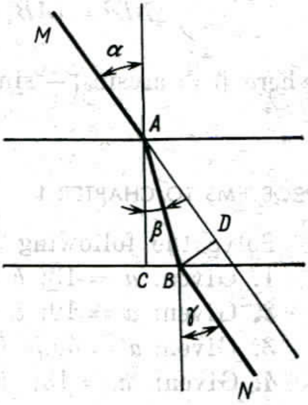


Fig. 18

According to the law of refraction of light, we have

$$\sin \alpha : \sin \beta = n, \quad \sin \beta : \sin \gamma = 1/n,$$

whence

$$\sin \beta = \frac{\sin \alpha}{n} \text{ and } \sin \beta = \frac{\sin \gamma}{n}.$$

Hence,

$$\sin \alpha = \sin \gamma, \text{ i.e. } \alpha = \gamma,$$

since the angles α and γ are acute.

The displacement of the beam due to refraction is equal to the distance between the parallel lines (MD) and (BN), i.e. equal to (BD), which is perpendicular to (MD).

The required quantity is the length of one of the sides of the right triangle ABD in which only one angle $\widehat{BAD} = \alpha - \beta$ is known, which is evidently insufficient for its solution. Therefore, let us first find the length of the hypotenuse $[AB]$, for which purpose we have to make use of the triangle ABC. In this triangle $|AC| = d$ and $\widehat{BAC} = \beta$. Hence, $|AB| = d/\cos\beta$.

Solving the triangle BAD, we finally find

$$|BD| = |AB| \sin \widehat{BAD} = \frac{d \sin(\alpha - \beta)}{\cos \beta},$$

where $\beta = \arcsin\left(\frac{1}{n} \sin \alpha\right)$.

PROBLEMS TO CHAPTER 1

Solve the following triangles (Problems **1** to **20**).

1. Given: $a = 13$; $b = 18$; $c = 15$. Find $\hat{A}$, $\hat{B}$, and $\hat{C}$.

2. Given: $a = 19$; $b = 34$; $c = 49$. Find $\hat{A}$, $\hat{B}$, and $\hat{C}$.

3. Given: $a = 45.5$; $b = 65$; $c = 10$. Find $\hat{A}$, $\hat{B}$, and $\hat{C}$.

4. Given: $a = 13$; $b = 15$; $\hat{C} = 59°30'$. Find c, $\hat{A}$, and $\hat{B}$.

5. Given: $b = 28$; $c = 42$; $\hat{A} = 124°$. Find a, $\hat{B}$, and $\hat{C}$.

6. Given: $a = 2.3$; $c = 1.7$; $\hat{B} = 29°50'$. Find b, $\hat{A}$, adn $\hat{C}$.

7. Given: b 22.5; $\hat{A} = 117°45'$; $\hat{B} = 20°40'$. Find a and c.

8. Given: $a = 218$; $\hat{B} = 56°51'$; $\hat{C} = 45°18'$. Find b and c.

9. Given: $a = 45$; $\hat{A} = 87°50'$; $\hat{B} = 10°50'$. Find b and c.

10. Given: $a = 460$; $b = 654$; $\hat{A} = 35°10'$. Find c, $\hat{C}$ and $\hat{B}$.

11. Given: $a = 13$; $b = 35$; $\hat{A} = 71°$. Find c, $\hat{B}$ and $\hat{C}$.

12. Given: $a = 19$; $b = 88$; $\hat{A} = 31°$. Find c, $\hat{B}$ and $\hat{C}$.

13. Given: $b = 360$; $c = 309$; $\hat{C} = 21°30'$. Find a, $\hat{A}$ and $\hat{B}$.

14. Given: $a + b = 76.2$; $\hat{A} = 15°28'$; $\hat{B} = 45°23'$. Find a, b, and c.

15. Given: $R = 7.92$; $\hat{A} = 113°17'$; $\hat{B} = 48°16'$. Find a, b, and c.

16. Given: $a + b = 489$; $\hat{A} = 70°$; $\hat{B} = 40°$. Find a, b, and c.

17. Given: $a - b = 23$; $\hat{A} = 108°$; $\hat{B} = 18°$. Find a, b, and c.

18. Given: $r = 5$, $\hat{A} = 22°$; $\hat{B} = 39°$. Find a, b, and c.

19. Given: $ab = 48$; $\hat{C} = 28°41'$; $\hat{A} = 115°40'$. Find a, b, c, and $\hat{B}$.

20. Given: $a = 32$; $b = 25$; $\hat{A} = 2\hat{B}$. Find c, $\hat{A}$, $\hat{B}$, and $\hat{C}$.

Reduce Problems **21** to **25** to one of the basic cases.

21. Given: the side a, adjacent angle $\hat{B}$, and the radius R of the circumscribed circle.

22. Given: two sides a and b and the ratio $c/\sin \hat{C} = q$.

23. Given: the angle $\hat{A}$, the side b and the ratio of the two remaining sides $a/c = p$.

24. Given: the angles $\hat{B}$ and $\hat{C}$, and the bisector l_a drawn from the vertex A.

25. Given: two sides a and b, and the bisector l_c drawn from the vertex C.

26. A chord common to two intersecting circles is seen from their centres at angles α and β. Find the radii of the circles if the distance between their centres is equal to a.

27. Two secants are drawn to a circle from an external point. One of them passes through the centre of the circle, its outside segment being equal to 1/4 of the radius of the circle. The outside and the inside segments of the other secant are equal to each other. Find the angle between the secants.

28. Taken on a side of the rhomb $ABCD$ is a point E which divides the side AB in the ratio 2 : 1. Find the angles of the triangle AEC if the acute angle $\hat{A}$ of the rhomb is equal to 72°.

29. The side of an isosceles triangle is equal to c and the vertex angle is equal to β. Find the radii of the circumcircle and the incircle.

30. Given in a right-angled triangle are the leg a and the adjacent acute angle α. Find the radii of the circumcircle and the incircle.

31. Given in a triangle ABC are the sides $|AB| = 10$, $|BC| = 8$, $|AC| = 6$. Find the radii of the circumcircle and the incircle.

32. Determine the angles of a right-angled triangle if the radius of the circumcircle is to the radius of the incircle as $5:2$.

33. In an isosceles trapezium the diagonal equal to d bisects its acute angle. Find the sides of the trapezium if its acute angle is equal to α.

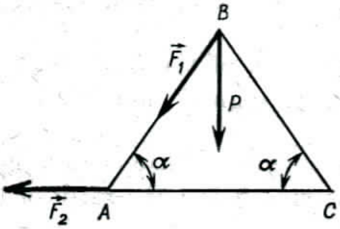


Fig. 19

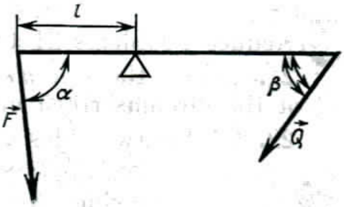


Fig. 20

34. A circle is circumscribed about a triangle whose angles $\hat{A}$ and $\hat{B}$ are given. Find the ratio of the perimeter of the triangle to the radius of the circumcircle.

35. The sides of a parallelogram are 7 cm and 11 cm long. Find its diagonals if one of them is 2 cm longer than the other.

36. The diagonals of a parallelogram are 24 cm and 28 cm long, and the difference between the sides is equal to 8 cm. Find the sides of the parallelogram.

37. The base of a parallelogram is 60 cm long, and the acute angle is equal to 48°. The diagonal of the parallelogram divides its obtuse angle in the ratio $1:3$. Find the perimeter of the parallelogram.

38. In a trapezium $ABCD$ the following elements are given: the bases $|AB| = a$, $|CD| = b$, and the angle $\widehat{BAD} = \alpha$. The angle between the diagonal $[BD]$ and the side $[AD]$ is equal to 2α. Find the nonparallel sides of the trapezium.

39. A material point is acted upon by the forces of 33 N and 39 N. Determine the angles between a balancing force of 60 N and each of the given forces.

40. A material point is kept in a state of equilibrium by the forces of 25 N, 36 N, and 29 N acting in one plane. Determine the angles between these forces.

41. Two forces of 36 N and 83 N act on a material point at an angle of $\alpha = 77°12'$. Determine the resultant force and the angle formed by it with the stronger force.

42. The rafters $[BA]$ and $[BC]$ (Fig. 19) form an angle α with the horizontal beam $[AC]$. A load of P N is suspended from B. Evaluate the force $\vec{F}_1$ pressing the rafters against the beam, and the force $\vec{F}_2$ stretching the beam.

43. A train moves at a speed of 12 m/s. A passenger sees the rain drops falling at an angle of 30° to the vertical. Determine the average velocity of the rain fall.

44. The arm l of a rectilinear lever is acted upon by a force of F N at an angle α to the lever, the other arm being acted upon by a force of Q N at an angle β to the lever (Fig. 20). Find the length of the second arm if the lever is kept in a state of equilibrium.

CHAPTER 2

LOGICAL STRUCTURE OF THE COURSE OF SOLID GEOMETRY

Plane geometry is the branch of geometry that treats of the properties and relations of plane figures, i.e. of sets whose points lie in one plane. Solid geometry studies figures in space (three dimensions), both plane and nonplanar, such as straight lines, polygons, circles, planes, polyhedrons, spheres, cylinders, etc. Therefore, we consider plane geometry as part of a more general course of solid geometry.

Sec. 8. Structure of the Course of Solid Geometry. Notation and Terminology

A systematic course of solid geometry is designed according to the same scheme as the course of plane geometry:

(1) basic geometric notions introduced without definition are listed;

(2) axioms describing the properties of the basic notions are formulated;

(3) using the basic notions, other geometric notions are defined;

(4) on the basis of axioms and definitions theorems are proved.

These are the basic notions in solid geometry: *point*, *straight line*, *plane*, and *distance*. We shall also use the *set*, which is a basic notion of all branches of mathematics.

The set of all points considered in solid geometry is called *space* and is denoted by the letter R. Any figure in solid geometry is considered as a set of points. Consequently, a figure which does not coincide with the entire space is its subset.

We shall denote points by the capital Latin letters A, B, C, . . ., straight lines by the lower-case Latin letters a, b, c, p, q, . . ., and planes by the Greek letters α, β, γ, ...

We shall use here the following symbols, terms and definitions borrowed from the set theory in geometry.

1. $A \in p$ = point A belongs to the straight line p, or point A lies on the line p, or the line p passes through the point A.

2. $A \in \alpha$ = point A belongs to the plane α, or point A lies in the plane α, or the plane α passes through the point A.

3. $p \subset \alpha$ = line p belongs to the plane α, or the line p lies in the plane α, or the plane α passes through the line p.

4. $A \notin p$, $A \notin \alpha$, $p \not\subset \alpha$ = point A does not belong to the line p (A does not lie on the line p), A does not lie in the plane α, the line p does not lie on or in the plane α.

5. (AB) = a line passing through the points A and B, $[AB]$ = a line segment with the end-points A and B, $\varnothing =$ = an empty set, i.e. a set containing no points.

6. $\cap$ = intersection, $\cup$ = join. For instance, $p \cap \alpha =$ $= A$, i.e. A is an intersection of the line p and the plane α; $p = \alpha \cap \beta$, i.e. the line p is an intersection of the planes α and β; $p \cap q = \varnothing$, i.e. the lines p and q does not intersect. $A \in \alpha \cap \beta$ means that the point A belongs to the intersection of the planes α and β, i.e. it lies in both planes.

A join of sets is defined as a set consisting of those and only those points which belong to at least one of the given sets. For example, $[AB] \cup [BC] \cup [CA]$ is the contour of the triangle ABC; $A \cup B$ is a union of two points A and B; $A \cup (AB) = (AB)$, since A lies on the line (AB); $A \cup B \cup$ $\cup\ C \subset p$ means that the points A, B, and C lie on one line p (i.e. they are collinear).

7. $A \Rightarrow B$ means that the validity of the condition A causes the validity of the assertion B. For instance, if one of the angles of a triangle is a right one, then the other two of its angles are acute. As is obvious, the converse is not always true.

8. $A \Leftrightarrow B$ means that the validity of B follows from the validity of A, and vice versa. For instance, if in a triangle the sum of two angles is equal to 90°, then such a triangle is right-angled. The condition "the sum of two angles of a triangle is equal to 90°" is equivalent to the condition "the triangle is right-angled": $(\hat{C} = 90°) \Leftrightarrow (\hat{A} + \hat{B} = 90°)$.

9. The generality quantifier $\forall$ is used as follows $\forall x$, meaning for each x, or for any x. For instance, $\forall$ natural n the number $n(n+1)$ is even; $\forall \triangle ABC$ the sum of its interior angles is equal to $180°$; $\forall A \in$ circle with centre at point O the following relation is true: $|OA| = r$, where r is the radius of the circle.

10. The existential quantifier $\exists$; $\exists x$ means "there exists x". For example, if $|a| \leqslant 1$, then there exists a solution for the equation $\sin x = a$. This can be written in a brief way: $\forall |a| \leqslant 1 \ \exists x$, satisfying the equation $\sin x = a$.

Sec. 9. Axioms of Belonging

Our course of solid geometry is based on the fourteen axioms formulated by the well-known Soviet academician Kolmogorov. These axioms are subdivided into five groups, the first group containing the *axioms of belonging*.

Axiom 1_1. *Any straight line p (and any plane α) is a non-empty set of points which does not coincide with the entire space R. There exists at least one straight line and at least one plane.*

Using the notation introduced, we may write this axiom as follows: $\exists$ straight line p and $\exists$ plane α, where $p \neq \varnothing$ and $p \neq R$, and also $\alpha \neq \varnothing$, and $\alpha \neq R$.

It follows from Axiom 1_1 that:

(1) since $p \neq \varnothing$, there exists at least one point belonging to p: $\exists A \in p$. But p does not coincide with the entire space, therefore there exists at least one point not belonging to p: $\exists B \notin p$;

(2) analogously, for the plane $\alpha: \exists C \in \alpha$ and $\exists D \notin \alpha$.

Axiom 1_2. *For any two different points A and B there exists one and only one straight line containing these points.*

Or briefly: $\forall A \neq B \exists$ the only straight line $p \supset A \cup B$. According to the terminology introduced above (Sec. 8, item 1), this axiom can be formulated using the "language" of geometry: *through any two different points there passes one and only one straight line.*

It follows from Axiom 1_2 that two different straight lines may have not more than one point in common.

Axiom 1_3. *For any three non-collinear points A, B, and C there exists one and only one plane containing these points.*

Briefly this axiom is written in the following way: $\forall A$, B, and C not lying on one line, $\exists$ the only plane $\alpha \supset A \cup B \cup C$. According to the terminology introduced in Sec. 8 (item 2), we may formulate this axiom using the "language" of geometry: *through any three points not lying on one straight line there passes one and only one plane.*

It follows from Axioms 1_1, 1_2, and 1_3 that space R contains not less than four points. Indeed, according to 1_1, in R there exist at least two points: $A \in p$ and $B \notin p$, $A \neq B$. According to 1_2 there exists a straight line $q \supset A \cup B$. Since $q \neq R$ (by virtue of Axiom 1_1), in R $\exists C \notin q$, i.e. $C \neq A$ and $C \neq B$. For three non-collinear points A, B, C $\exists$ plane $\alpha \supset A \cup B \cup C$ (by Axiom 1_3), and $\alpha \neq R$ (by Axiom 1_1). Therefore in R $\exists$ point $D \notin \alpha$, i.e. $D \neq A$, $D \neq B$, $D \neq C$. So, there exist four different points in R.

The following two axioms establish relations between a straight line and a plane.

Axiom 1_4. *If two different points A and B belong to the plane α, then all points of the straight line passing through the points A and B also belong to α.*

This is how this axiom is written in a brief form: $(A \cup B \subset \alpha) \Rightarrow ((AB) \subset \alpha)$. According to the terminology introduced in Sec. 8 (item 3), let us formulate this axiom using the "language" of geometry: *a straight line passing through two different points of a plane is contained in this plane.*

Axiom 1_5. *If two different planes* α and β *have a point A in common, then their intersection is a straight line passing through the point A.*

The brief notation of this axiom: $(A \in \alpha \cap \beta) \Rightarrow (\alpha \cap \beta = p \supset A)$.

Let us give some examples of such sets (spaces), in which some of the mentioned axioms are fulfilled. In mathematics such examples are called "models".

Example 1. The space R consists of two points A and B: $R = \{A, B\}$. Each of the points is both a straight line and a plane: $p_1 = \alpha = \{A\}$, $p_2 = \beta = \{B\}$. As is obvious, in this space Axiom 1_1 is fulfilled, while Axioms 1_2 and 1_3 are not.

Example 2. The space R consists of three points A, B, and C: $R = \{A, B, C\}$. Sets consisting of one point will be called "straight lines": $p_1 = \{A\}$, $p_2 = \{B\}$, $p_3 = \{C\}$, and those consisting of two points we shall call "planes": $\alpha_1 = \{A, B\}$, $\alpha_2 = \{A, C\}$, $\alpha_3 = \{B, C\}$. In this space, as can be easily checked, only Axioms 1_1 and 1_5 are fulfilled.

Example 3. A space consists of four points A, B, C, and D: $R = \{A, B, C, D\}$. In this space the sets consisting of two points will be called "straight lines": $p_1 = \{A, B\}$, $p_2 = \{A, C\}$, $p_3 = \{A, D\}$, etc. The sets consisting of three points will be called "planes": $\alpha_1 = \{A, B, C\}$, $\alpha_2 = \{A, B, D\}$, $\alpha_3 = \{A, C, D\}$, $\alpha_4 = \{B, C, D\}$. It is easy to check that the axioms 1_1 to 1_5 are fulfilled in this space.

Example 4. Space consists of eight points: $R = \{A, B, C, D, E, H, K, M\}$. In this case each pair of points is called a "straight line", and any three points represent a "plane". For this space Axioms 1_1 to 1_4 are fulfilled, and Axiom 1_5 is not fulfilled. Indeed, two "planes", for instance, $\{BCH\}$ and $\{CDK\}$, having a common point C, have no "straight line" in common.

The existence of models for which the above mentioned axioms are fulfilled prove their consistency.

Sec. 10. Axioms of Distance

Three *axioms of distance* constitute the second group of axioms. They will help us introduce very important notions: "a point lies between two other points" and the "line segment".

Axiom 2_1. *For any two points A and B of space R there exists a non-negative number denoted $\rho(A, B)$ and called the distance from A to B. This distance is equal to zero if and only if the points A and B coincide.*

Using the symbols introduced, we may write this axiom in the following way: $\forall A \in R$ and $B \in R$ $\exists$ number $\rho(A, B) \geqslant 0$, and $(\rho(A, B) = 0) \Leftrightarrow (A = B)$.

In this axiom the points A and B are not equal. The following axiom eliminates this "defect".

Axiom 2_2. *The distance from the point A to the point B is equal to the distance from the point B to the point A: $\rho(A, B) = \rho(B, A)$.*

We may now speak simply of the distance between two points A and B.

Axiom 2_3 (the triangle inequality). *For any three points A, B, and C of the space R the distance between any pair of points does not exceed the sum of the distances between the* *other two pairs, i.e.* $\forall A$, B, *and* $C \subset R$

$$\rho(A, C) \leqslant \rho(A, B) + \rho(B, C);\ \rho(A, B) \leqslant \rho(A, C) + \rho(B, C);$$

$$\rho(B, C) \leqslant \rho(A, C) + \rho(A, B). \tag{1}$$

Corollary. *For any three points A, B, and C of the space R the distance between any pair of points is not less than the difference of the distances between the other two pairs:*

$$\rho(A, C) \geqslant \rho(A, B) - \rho(B, C);\ \rho(A, B) \geqslant \rho(A, C) - \rho(C, B);$$

$$\rho(B, C) \geqslant \rho(B, A) - \rho(A, C). \tag{2}$$

The proof of this corollary is obvious.

Joining the inequalities (1) and (2), we get the following inequality

$$\rho(A, B) - \rho(B, C) \leqslant \rho(A, C) \leqslant \rho(A, B) + \rho(B, C)$$

and two similar ones.

Definition 1. A point B *lies between* the points A and C if these three points are different and

$$\rho(A, C) = \rho(A, B) + \rho(B, C). \tag{3}$$

Definition 2. A set consisting of two different points A and B and all points lying between them is called the *line segment* $[AB]$.

The points A and B are called its *end-points*; the points lying between A and B are said to be *internal* points of the line segment.

Example. In the space $R = \{A, B, C, D\}$ considered in Example 3 of Sec. 9, let us introduce the distance between two points according to the following rule:

$$\rho(A, B) = \rho(A, C) = \rho(A, D) = \rho(B, C) = \rho(B, D) = \rho(C, D) = 1,$$

$$\rho(A, A) = \rho(B, B) = \rho(C, C) = \rho(D, D) = 0$$

We illustrated above that this space (a set of two points is a "straight line", a set of three points is a "plane") satisfies all the axioms of the first group. It is obvious that the introduced number ρ satisfies Axioms 2_1 and 2_2. Let us check whether Axiom 2_3 is fulfilled. For three different points the triangle inequality is fulfilled, since $1 \leqslant 1 + 1$. If the three points coincide, then Axiom 2_3 is also true ($0 \leqslant 0 + 0$). If, finally, two of the three points coincide, then we get either $0 \leqslant 1 + 0$, or $1 \leqslant 1 + 0$. Thus, in the set under consideration all axioms of the first and the second groups are fulfilled. But in this space none of the points lies between other two points, since for three different points the equation (3) is not fulfilled. Hence, there is no line segment in this space.

Sec. 11. Axioms of Order

Axioms belonging to the first and second groups are not sufficient for establishing a complete course of solid geometry. The example considered in Sec. 10 shows that the spaces generated by these axioms are very "poor", i.e. they may consist of a finite number of points (four in the mentioned example) and have even no line segment. Moreover, using axioms only of the first and second groups, we are unable to introduce the notion of the ray, half-plane, angle, etc. All these notions are introduced with the aid of the *axioms of order.*

Axiom 3_1. *Any point O of a straight line p divides the set of all points of this line which differ from O into two nonempty sets (OM) and (ON) according to the following rule:*

(1) *points A and B belong to different sets if O lies between them, i.e.* $\rho(A, O) + \rho(O, B) = \rho(A, B)$;

(2) *points A and B belong to one set if one of them lies between the point O and the other point, i.e.* $\rho(O, A) = \rho(O, B) + \rho(B, A)$, or $\rho(O, B) = \rho(O, A) + \rho(A, B)$.

Each of these sets is called the *ray*. A ray is said to be *open* if it does not contain the point O; if it contains O, then it is said to be *closed*. The point O is called the *initial* point or the *origin* of the ray. Often a ray with or without its initial point is called a half-line.

Thus, each point of a straight line divides this line into two rays with the origin at this point. A closed ray (simply ray) contains the origin and an open ray does not.

Example. Prove that of three different points lying on one straight line one is found between two others.

Solution. Let us take one of the given points, say, point A. According to Axiom 3_1, the point A divides the straight line into two rays. If the points B and C lie on different rays, then this means that A lies between them. And if B and C belong to one ray, then either B lies between A and C, or C between A and B.

Axiom 3_2. *For any non-negative number a on a given ray with the origin O there exists one and only one point A whose distance from the point O is equal to a, i.e.* $\forall a \geqslant 0\ \exists A \in (OM)$ *so that* $\rho(O, A) = a$; $\forall B \in (OM)$ *and* $B \neq A \rho(O, B) \neq a$.

It follows from this axiom that:

(1) the ray contains an infinite set of points;

(2) on the given straight line containing the point O, for a given distance $a > 0$ there are exactly two points lying at a distance a from the point O.

It follows from Axiom 3_1 and the axioms of distance (see the example) that of any three collinear points one lies between two others. Is the converse true?

The following axiom gives the answer to this question.

Axiom 3_3. *If a point C lies between points A and B, then the points A, B, C belong to one straight line, i.e. if* $\rho(A, B) = \rho(A, C) + \rho(C, B)$, *then* $\exists p \supset A \cup B \cup C$.

Two corollaries follow from this axiom and the axioms of the first and second groups.

Corollary 1. *The line segment $[AB]$ is a subset of the straight line (AB).*

In fact, (AB) is the only straight line passing through the points A and B (Axiom 1_2). For any point $C \in [AB]$ the following equation is true: $\rho(A, C) + \rho(C, B) = \rho(A, B)$, which means that (Axiom 3_3) A, B, and C are collinear points, i.e. they lie on one straight line which, obviously, coincides with (AB). Thus, any point of the line segment $[AB]$ belongs to the straight line (AB).

Corollary 2. *For three points not lying on one straight line the following inequality is fulfilled:* $\rho(A, B) < \rho(A, C) + \rho(C, B)$.

Indeed, for any three points A, B, C the inequality $\rho(A, B) \leqslant \rho(A, C) + \rho(C, B)$ is fulfilled. If $\rho(A, B) = \rho(A, C) + \rho(C, B)$, then, according to Axiom 3_3, the points A, B, C are collinear, which is false. Consequently, $\rho(A, B) < \rho(A, C) + \rho(C, B)$.

Hence, the equation $\rho(A, B) = \rho(A, C) + \rho(C, B)$ is the necessary and sufficient condition of belonging of three points to one straight line. Using it, we may formulate Axiom 1_3 in the following way: $\forall$ points A, B, C such that $\rho(A, B) < \rho(A, C) + \rho(C, B)$, $\exists$ the only plane $\alpha \supset A \cup B \cup C$.

Axiom 3_4. *Any straight line divides the set of not belonging to it points of plane α into two non-empty sets α_1 and α_2 according to the following rule*:

(1) *points A and $B \not\subset p$ belong to one set (α_1 or α_2), if the line segment $[AB]$ does not intersect the straight line p*;

(2) *points A and $B \not\subset p$ belong to different sets if the line segment $[AB]$ intersects the straight line p.*

In the first case the points A and B are said to be *not separated* by the straight line p, in the second case they are *separated* by this line.

The sets α_1 and α_2 obtained as a result of dividing the plane α by the straight line p are called *open half-planes*. An open half-plane with the straight line p adjoined to it is called a *closed half-plane* (or simply *half-plane*).

Note. According to Axiom 1_1, a straight line does not coincide with the entire space. A stronger statement follows from Axiom 3_4: *for any straight line p contained in plane α there exist points in this plane not belonging to p, i.e. the straight line does not coincide with the entire plane α.*

The notions of the ray and half-plane were introduced with the aid of the axioms of the third group. The same axioms are sufficient to introduce the notion of half-space.

Definition. The points A and B of space R not contained in plane α are said to be *separated* by this plane if the line segment $[AB]$ intersects α, i.e. $\alpha \cap [AB] \neq \varnothing$. The points A and B are *not separated* by the plane α if $[AB]$ does not intersect α, i.e. $\alpha \cap [AB] = \varnothing$.

Thus, the plane α divides the space R into two halves R_1 and R_2 called *open half-spaces*: the points A and B not lying on α belong to one half-space if they are not separated by the plane α. They belong to different half-spaces if they are separated by the plane α.

Let us show that these half-spaces are not empty. According to Axiom 1_1, in R $\exists A \notin \alpha$. We take an arbitrary point O on the plane α; by Axiom 1_2, the points O and A determine the straight line (OA). The point O divides this line into two rays. It is obvious, that all points lying on one ray (there is an infinite set of them) belong to one half-space, and the points lying on different rays belong to different half-spaces.

An open half-space with the plane α adjoined to it is called a *closed half-space* (or simply a *half-space*).

Sec. 12. Axiom of Plane Mobility

Here are some examples of displacing a plane: identity mapping, translation, rotation, axial and central symmetry.

To specify an onto mapping of a plane it is sufficient to supply each point X with its image $f(X)$. The images may be chosen arbitrarily, provided the following condition is satisfied: $f(X) \neq f(Y)$ for $X \neq Y$.

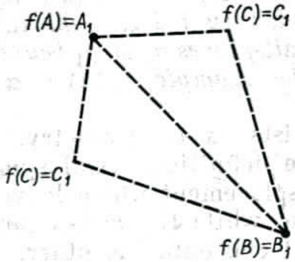


Fig. 21

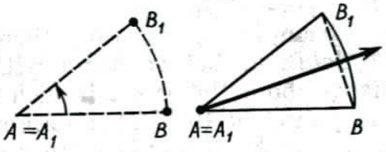


Fig. 22

Displacement keeps distances unchanged, therefore the images of points are related by the condition $\rho(f(X), f(Y)) = \rho(X, Y)$, whence it follows that if there exists a displacement which maps the points A and B into $A_1 = f(A)$ and $B_1 = f(B)$, where $\rho(A_1, B_1) = \rho(A, B)$, then either C_1 or C_2 may be the image of a third point C, since the distances $f(C)$ from the points A_1 and B_1 must be equal to $\rho(A, C)$ and $\rho(B, C)$, respectively (Fig. 21). It means that if such a displacement $A_1 = f(A)$, $B_1 = f(B)$ exists, then it is not unique.

Let us look for displacements in which the point A remains fixed, and B is mapped into B_1, i.e. $f(A) = A$, $f(B) = B_1$, where $\rho(A, B) = \rho(A, B_1)$ (Fig. 22).

As is obvious, rotation about the centre A through an angle BAB_1 and axial symmetry with the axis passing through the point A and perpendicular to BB_1 are also examples of such displacement. The question is: How many displacements leave the point A immovable and map the point B into the point B_1?

Thus, there arise two questions:

(1) Is there a displacement which maps two arbitrarily given points A and B onto arbitrary points A_1 and B_1 satisfying the only condition $\rho(A_1, B_1) = \rho(A, B)$?

(2) How many displacements are possible if only images of two points are indicated?

The answer to both of them is given by the *axiom of plane mobility* constituting the fourth group of axioms.

Axiom 4. *If the points A, B, A_1, B_1 belong to the plane α and $\rho(A, B) = \rho(A_1, B_1)$, then there exist two and only two displacements f_1 and f_2 bringing A into A_1, and B into B_1. In this case the half-plane α bounded by the straight line (AB) will be mapped onto one of the half-planes α_1 or β_1 bounded by the straight line (A_1B_1): if, for example, $f_1(\alpha) = \alpha_1$, then $f_2(\alpha) = \beta_1$.*

Example. Prove that there exists axial symmetry.

Solution. Let us begin with the definition: axial symmetry about the axis l is such a displacement which leaves the points of the line l immovable, while the half-planes separated by the axis l are mapped one onto the other.

The straight line l divides the plane into two half-planes α and β. Let us take on l two arbitrary points $A \neq B$. According to Axiom 4, there exist two and only two displacements f_1 and f_2 bringing the point A into A_1 and the point B into B_1. One of these displacements is the identity transformation E, i.e. $f_1 = E$. Hence, $f_1(\alpha) = \alpha$, and then $f_2(\alpha) = \beta$.

To complete the proof let us show that f_2 leaves the points of the line l in their places. Suppose that C is any point of l different from A and B. One of these points lies between the two others, for instance, C between A and B. Then (see Sec. 11)

$$\rho(A, B) = \rho(A, C) + \rho(C, B),$$

or

$$\rho(A, B) = \rho(f_2(A), f_2(C)) + \rho(f_2(C), f_2(B)).$$

Since $f_2(A) = A$ and $f_2(B) = B$, we have $\rho(A, B) = \rho(A, f_2(C)) + \rho(f_2(C), B)$, i.e. $f_2(C)$ lies on the line l

between A and B. If $C \geqslant f_2(C)$, then

$$\rho(f_2(C), C) = \rho(A, C) - \rho(A, f_2(C)) =$$
$$= \rho(A, C) - \rho(f_2(A), f_2(C)) = \rho(A, C) - \rho(A, C) = 0.$$

Consequently, $f_2(C) = C$. And if $C \leqslant f_2(C)$, then $\rho(C, f_2 \times \times (C)) = \rho(A, f_2(C)) - \rho(A, C) = 0$ and again $C = f_2(C)$

Sec. 13. Axiom of Parallelism

The fifth group of axioms consists of the *axiom of parallelism*.

Axiom 5. *If a point A and a straight line p lie in a plane α, then in this plane there passes through the point A at most one straight line parallel to the given line p.*

This axiom emphasizes that the point A, the straight line p, and the line parallel to it lie in one and the same plane.

The statement on uniqueness of a straight line passing through a given point in space parallel to the given line is a theorem. Its validity follows as a corollary from this and other axioms.

Definition 1. Two lines p and q in space are called *parallel* if there is a plane in which they both lie, and either they coincide, or do not intersect.

Definition 2. A straight line p *is parallel to a plane* α if p either lies in this plane, or has with it no points in common.

Example. The space R consists of four points: $R = \{A, B, C, D\}$. Any pair of the points is called a "straight line", any three points a "plane". Are there parallel "lines" in this space?

Solution. By definition, each straight line is parallel to itself. Are there different pairs of parallel "lines" in this space?

Any two non-intersecting "lines" do not belong to one "plane", and, consequently, they are not parallel. Any two "lines" lying in one "plane" intersect.

Hence, there are no different parallel "lines" in the given space. By the way, let us point out that any "plane" of this space does not contain different parallel "lines" either.

Sec. 14. Corollaries Following from the Axioms

Corollary 1. *Through a straight line p and a point A not on the line only one plane can be drawn.*

Given: a line p and a point $A \notin p$.

Prove: $\exists$ the only plane $\alpha \supset A \cup p$.

Proof. There are two different points B and C on the line p. Three non-collinear points A, B, and C specify the unique plane α, containing these points (Axiom 1_3). Any other plane containing p and A contains A, B, and C and therefore coincides with the plane α, i.e. the plane α is unique.

Corollary 2. *Through two intersecting lines only one plane can be drawn.*

Given: two straight lines p and q, $p \neq q$, $p \cap q \neq \varnothing$.

Prove: $\exists$ the unique plane $\alpha \supset p \cup q$.

Proof. Let us denote the point of intersection of p and q by A. On the line p $\exists B \neq A$, and on the line q $\exists C \neq A$. Three points A, B, and C not lying on one straight line, determine the unique plane α, which contains these points. Since A and B belong to α, then also $p \subset \alpha$ (Axiom 1_2); since A and C belong to α, then $q \subset \alpha$. Any other plane containing p and q contains the points A, B, C, and therefore coincides with α.

Corollary 3. *Through two different parallel straight lines it is possible to draw only one plane.*

Given: $p \parallel q$, $p \neq q$.

Prove: $\exists$ the unique plane $\alpha \supset p \cup q$.

Proof. By the theorem, straight lines p and q lying in space are parallel. Consequently, they belong to some plane α (by the definition of parallel lines in space, see Sec. 13).

Let us show that this plane is unique. To do this we take two points $A \neq B$ on the line p, and a point C on the line q. Any plane containing p and q must also contain the points A, B, C which are not collinear, and such a plane is unique.

Corollary 4. *Through any point in space there passes one and only one straight line parallel to the given line.*

Given: a straight line p, and a point A.

Prove: $\exists$ the unique line $q \parallel p$.

Proof. If $A \in p$, then $q = p$. Suppose $A \notin p$. Through the point A and line p there passes the unique plane α (Corollary 1). In this plane there exists the unique straight line passing through A and parallel to p. Any other straight line passing through the point A parallel to p must lie in one plane with them, i.e. in the plane α. Hence, it coincides with q.

Note. While proving Corollary 4, we also indicated the method of constructing a straight line in space parallel to a given one.

CHAPTER 3

PARALLELISM IN SPACE

Sec. 15. The Relative Position of Two Straight Lines in Space. The Relative Position of a Plane and a Straight Line in Space

Two straight lines in space may:

(1) lie in one plane and

(a) have only one point in common: $p \cap q = A$ (the lines intersect);

(b) have no points in common (the lines are parallel);

(c) coincide (the lines are parallel);

(2) not lie in one plane (the lines are skew).

Definition. Straight lines p and q are said to be *skew* if there is no plane containing both of them.

Hence, two straight lines in space either intersect, or not. In the latter case they are parallel or skew.

Problem. Given a straight line p. Find a straight line q which does not intersect p, nor is parallel to it (i.e. p and q must be skew).

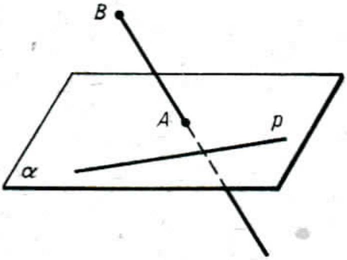


Fig. 23

Solution. By Axiom 1_1 $\exists A \notin p$, and by virtue of Corollary 1 of Sec. 14, $\exists \alpha \supset p \cup A$ (Fig. 23). According to the same axiom $\exists B \notin \alpha$. Consequently, the straight line $q = (AB)$ does not belong to α either. Moreover, there is no plane containing p and q. Indeed, any other plane $\beta \supset p \cup q$ must contain p and A simultaneously, i. e. must coincide with α (Corollary 1 of Sec. 14), which is false, since $B \not\subset \alpha$. Hence, p and q are skew lines. As is obvious, the problem has an infinite number of solutions.

A straight line p and a plane α may have:

(1) only one common point: $p \cap \alpha = A$ (the line p cuts the plane α at point A, which is called the *trace* of the line on the plane α);

(2) more than one common point (then, according to Axiom 1_4, $p \subset \alpha$, i.e. p is parallel to α);

(3) no points in common (p is parallel to α).

Thus, a straight line and a plane in space either intersect, or are parallel to each other.

Sec. 16. Parallelism of a Straight Line and a Plane

Theorem 1. *A straight line p parallel to the straight line q is parallel to any plane passing through q.*

Given: $p \parallel q$, $q \subset \alpha$.

Prove: $p \parallel \alpha$.

Proof. If $\alpha \supset p$, then $p \parallel \alpha$. Suppose $p \not\subset \alpha$ (Fig. 24). Let us draw $\beta \supset p \cup q$ (see Corollary 3 of Sec. 14). Since $q \subset \alpha$ (by hypothesis) and $q \subset \beta$ (by construction), then $q = \alpha \cap \beta$. Let us assume that the theorem is not true, i.e. $p \nparallel \alpha$. Then $\exists C = p \cap \alpha$ (Fig. 24). Since $C \in p$, and $p \subset \beta$, then $C \in \beta$ and, consequently, $C \in \alpha \cap \beta = q$. Hence, $C \in p \cap q$ which is false, since $p \parallel q$.

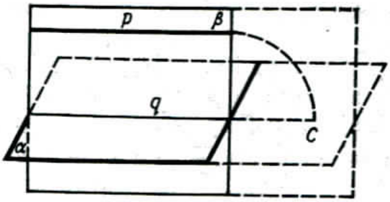


Fig. 24

Theorem 2. *If a plane β passes through a line p parallel to the plane α and intersects α, then the line of their intersection $q = \alpha \cap \beta$ is parallel to p.*

Given: $p \subset \beta$, $p \parallel \alpha$, $q = \alpha \cap \beta$.

Prove: $p \parallel q$.

Proof. Suppose that $p \nparallel q$. Then $\exists C = p \cap q$ (Fig. 25). Since $C \in q$, and $q \subset \alpha$, then $C \in \alpha$. Hence, $C \in p \cap \alpha$, which is not true, since $p \parallel \alpha$.

Theorem 3. *A straight line p, which is parallel to each of two intersecting planes α and β, is parallel to the line of their intersection* (Fig. 26).

Given: $p \parallel \alpha$, $p \parallel \beta$, $q = \alpha \cap \beta$.

Prove: $p \parallel q$.

Proof. If $p \subset \alpha$ or $p \subset \beta$, then it follows from Theorem 2 that $p \parallel \alpha \cap \beta = q$.

Suppose $p \not\subset \alpha$ and $p \not\subset \beta$. Let us draw a plane γ through the line p and a point $A \in \alpha \cap \beta$. According to Theorem 2, $p \parallel \alpha \cap \gamma$ and $p \parallel \beta \cap \gamma$, i.e. p is parallel to the two lines passing through a common point A, whence it follows that these lines coincide: $\alpha \cap \gamma = \beta \cap \gamma = q_1$. Consequently, $p \parallel q_1$. Since $q_1 \subset \alpha$ and $q_1 \subset \beta$, we have $q_1 = \alpha \cap \beta = q$. Hence, $p \parallel q$.

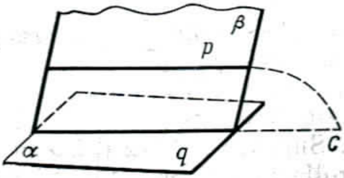


Fig. 25

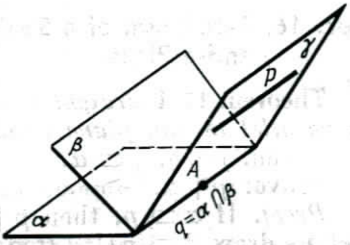


Fig. 26

Sec. 17. Parallel Planes

Definition. Two planes are called *parallel* if they have no points in common, or if they coincide.

Thus, two planes in space either intersect, or they are parallel to each other.

Theorem 1 (the test for parallelism of two planes). *If two intersecting lines contained in plane α are respectively parallel to two lines contained in plane β, then these planes are parallel.*

Given: $p \cup q \subset \alpha$, $p \nparallel q$, $p_1 \cup q_1 \subset \beta$, $p_1 \parallel p$, $q_1 \parallel q$ (Fig. 27).

Prove: $\alpha \parallel \beta$.

Proof. Suppose the opposite, i.e. $\alpha \nparallel \beta$. Then α and β intersect along a straight line $l = \alpha \cap \beta$. Since $p \parallel p_1$ and

$q \parallel q_1$, we have $p \parallel l$ and $q \parallel l$ (according to Theorem 2 of Sec. 16). Thus, from the supposition $\alpha \nparallel \beta$ it follows that two intersecting lines p and q are parallel to one and the same line l, which is false. Hence, $\alpha \parallel \beta$.

Theorem 2. *Straight lines p and q obtained in intersection of two parallel planes α and β by a third plane γ are parallel to each other.*

Given: $\alpha \parallel \beta$, $p = \alpha \cap \gamma$, $q = \beta \cap \gamma$ (Fig. 28).

Prove: $p \parallel q$.

Proof. If $\alpha = \beta$, then $p = q$ and, hence, $p \parallel q$. Suppose now that $\alpha \cap \beta = \varnothing$. The lines p and q lie in one plane γ, therefore either $p \parallel q$, or $p \cap q = C$. If $p \cap q = C$, then $C \in p \subset \alpha$ and $C \in q \subset \beta$, i.e. the planes α and β have a common point C, which is false, since $\alpha \parallel \beta$. Hence, $p \parallel q$.

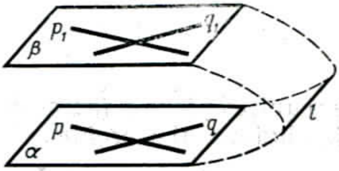


Fig. 27

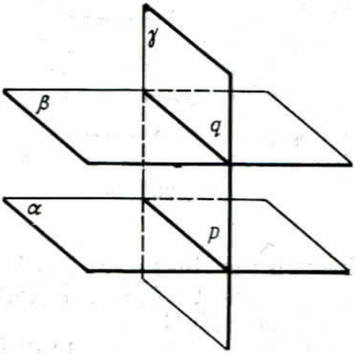


Fig. 28

Theorem 3. *The segments of parallel lines contained between parallel planes have equal lengths.*

Given: $\alpha \parallel \beta$, $p \parallel q$, $A = \alpha \cap p$, $B = \alpha \cap q$, $C = \beta \cap p$, $D = \beta \cap q$ (Fig. 29).

Prove: $|AC| = |BD|$.

Proof. Let us produce a plane γ through the parallel lines p and q. This plane will intersect the planes α and β along parallel lines $(AB) = \alpha \cap \gamma$ and $(CD) = \beta \cap \gamma$ (according to Theorem 2). Since $(AB) \parallel (CD)$, and $(AC) \parallel (BD)$, the figure $ABCD$ is a parallelogram and, consequently, $|AC| = |BD|$.

Problem 1. Through the point A draw a plane β parallel to a given plane α.

Solution. If $A \in \alpha$, then the required plane $\beta = \alpha$. Let $A \notin \alpha$. In the plane α we draw two intersecting lines p and q (Fig. 30), and then from the point A we construct $p_1 \parallel p$ and $q_1 \parallel q$ (see Corollary 4 of Sec. 14). The plane passing through the lines p_1 and q_1 is the required plane β (according to Theorem 1 of Sec. 17). As is obvious, the problem has a unique solution.

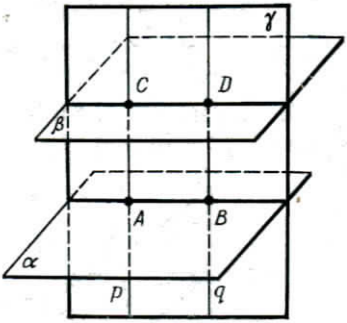


Fig. 29

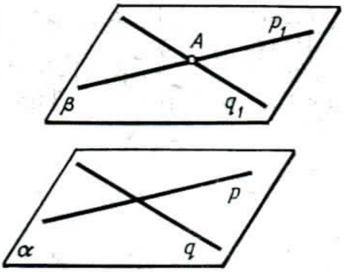


Fig. 30

Thus, we have obtained another important result: *through a given point it is possible to draw one and only one plane parallel to a given plane.*

Problem 2. Through the line p draw a plane α parallel to the given line l.

Solution. Let us consider three cases:

(1) $p \parallel l$. Then any plane passing through p is parallel to l (according to Theorem 1 of Sec. 16). The problem has an infinite number of solutions;

(2) the lines p and l intersect. Then the plane α passing through p and l contains l and, consequently, is parallel to l. In this case the solution is unique.

(3) p and l are skew lines. Let us take an arbitrary point $A \in p$ (Fig. 31) and, through it, draw a straight line q parallel to l (see Corollary 4 of Sec. 14). Two intersecting lines p and q specify the unique plane α. Since $l \parallel q$ and $q \subset \alpha$, we have $l \parallel \alpha$ (according to Theorem 1 of Sec. 14). Prove that in this case the problem also has only one solution.

Sec. 18. Direction in Space. The Angle Between Two Lines

Theorem 1. *Two straight lines p and l separately parallel to a third line q are parallel to each other.*

Given: $p \parallel q$, $l \parallel q$.

Prove: $p \parallel l$.

Proof. Let us take an arbitrary point $A \in p$ (Fig. 32) and draw a plane $\alpha \supset A \cup q$ and a plane $\beta \supset A \cup l$. Since $l \parallel q$, and $q \subset \alpha$, we get $l \parallel \alpha$ (according to Theorem 1 of Sec. 16) and, hence, $l \parallel \alpha \cap \beta = p_1$ (according to Theorem 2 of Sec. 16).

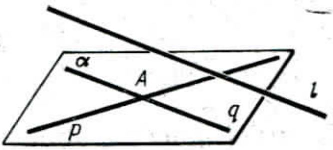


Fig. 31

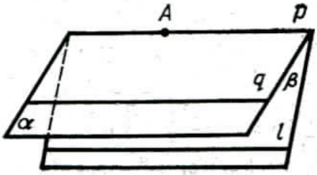


Fig. 32

Let us now show that $p_1 = p$. Since $q \parallel l$, then $q \parallel \beta$, and therefore $q \parallel \alpha \cap \beta = p_1$. Thus, $q \parallel p_1$ and $q \parallel p$, where $p_1 \cap p \ni A$. Hence, $p = p_1$ and $l \parallel p$, which was required to be proved.

It follows from Theorem 1 that parallelism of straight lines has the transitive property: if $p \parallel q$, and $q \parallel l$, then $p \parallel l$.

Definition 1. The set of all the lines parallel to one and the same line is called the *sheaf* of parallel lines.

It follows from the transitive property that any two sheaves of parallel lines are parallel to each other.

Let us consider two rays $h_1 = [O_1A_1)$ and $h_2 = [O_2A_2)$ lying on different parallel lines p and q (Fig. 33). The straight line (O_1O_2) divides the plane α containing the lines p and q into two half-planes α_1 and α_2.

Definition 2. The rays h_1 and h_2 lying on different parallel lines are said to be *co-directed* ($h_1 \uparrow\uparrow h_2$), if they lie in one half-plane bounded by the straight line passing through the origin of the rays (Fig. 33); the rays h_1 and h_2 are said to be *oppositely directed* ($h_1 \uparrow\downarrow h_2$), if they lie in different half-planes (Fig. 34).

If the rays h_1 and h_2 lie on one line, then $h_1 \uparrow\downarrow h_2$ if $h_1 \cap h_2$ is a line segment (Fig. 35), and $h_1 \uparrow\uparrow h_2$ if $h_1 \cap h_2$ is a ray (Fig. 36).

As is obvious, if $h_1 \uparrow\uparrow h_2$ and $h_2 \uparrow\uparrow h_3$, then $h_1 \uparrow\uparrow h_3$.

The set of all co-directed rays emanated from each point of space is said to specify a definite direction in space. Since co-direction of rays has the transitive property, direction in space is defined by specifying any ray drawn from any point of the space.

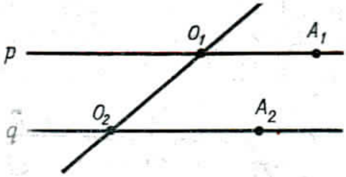


Fig. 33

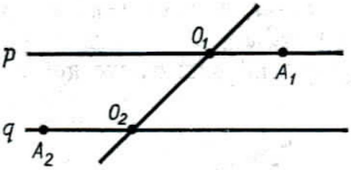


Fig. 34

Let two rays $[OA)$ and $[OB)$ emanate from the point O. If $[OA) \uparrow\uparrow [OB)$, then these rays specify one direction, and we then suppose that $(\widehat{[OA), [OB)}) = 0°$. If $[OA)$ and $[OB)$ are not co-directed, then these rays specify different directions. The *angle between these directions* is defined as the magnitude of the convex angle bounded by the rays $[OA)$ and $[OB)$. Obviously, in this case $0° < (\widehat{[OA), [OB)}) \leqslant 180°$, where $(\widehat{[OA), [OB)}) = 180°$, if $[OA) \uparrow\downarrow [OB)$, i.e. if the rays specify opposite directions.

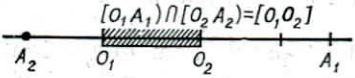


Fig. 35

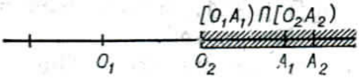


Fig. 36

Let us show that the angle between the directions is independent of the choice of point O. To this purpose we are going to prove the following theorem.

Theorem 2. *Two convex angles with respectively co-directed sides are congruent.*

Given: $[OA) \uparrow\uparrow [O_1A_1)$, $[OB) \uparrow\uparrow [O_1B_1)$.

Prove: $\angle AOB \cong \angle A_1O_1B_1$ (Fig. 37).

Proof. Through the intersecting lines (OA) and (OB) we draw a plane α, and through the lines (O_1A_1) and (O_1B_1) a plane β. According to Theorem 1 of Sec. 17, $\alpha \parallel \beta$. We then lay off congruent segments $[OA] \cong [OB] \cong [O_1A_1] \cong \cong [O_1B_1]$ on the sides of the angles AOB and $A_1O_1B_1$, and draw straight lines (AA_1), (BB_1), (OO_1), (AB), and (A_1B_1).

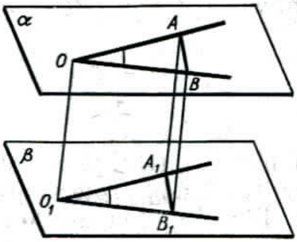


Fig. 37

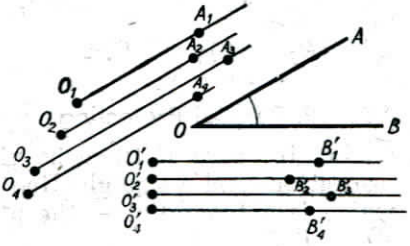


Fig. 38

The quadrilaterals OO_1A_1A and OO_1B_1B are parallelograms, therefore $(OO_1) \parallel (AA_1)$, $(OO_1) \parallel (BB_1)$, and hence, $|AA_1| = = |BB_1|$ (by Theorem 3 of Sec. 17). Whence it follows that AA_1B_1B is a parallelogram, and $|AB| = |A_1B_1|$. Thus, the isosceles triangles AOB and $A_1O_1B_1$ are congruent, and hence the angles AOB and $A_1O_1B_1$ are congruent.

Thus, the angle between two directions is defined as the magnitude of a convex angle bounded by any two rays of these directions emanating from one arbitrarily taken point in space (see Fig. 38).

Intersecting lines determine two pairs of vertical angles. The magnitude of the smaller angle is called the *angle between the given lines*. If the lines are parallel, then the angle between them is considered to be equal to zero.

The *angle between skew lines* is defined as the angle between two intersecting lines respectively parallel to the given skew lines. Hence, $0° \leqslant \widehat{(p, q)} \leqslant 90°$.

Straight lines p and q are said to be *mutually perpendicular* $(p \perp q)$ if the angle between them is equal to 90°, i.e.

$\widehat{(p, q)} = 90^\circ$. Perpendicular lines either intersect, or represent skew lines.

Example. Consider the cube $ABCDA_1B_1C_1D_1$ (Fig. 39). As is obvious, (AC) and (B_1D_1) are skew lines. Since $[AC) \uparrow\uparrow [A_1C_1)$, and $[B_1D_1) \uparrow\uparrow [B_1D_1)$, then $\widehat{((AC), (B_1D_1))} = \widehat{((A_1C_1), (B_1D_1))} = 90^\circ$, i.e. (AC) is perpendicular to (B_1D_1). (BC) and (A_1C_1) are also skew lines. Since $[AC) \uparrow\uparrow [A_1C_1)$, and $[BC) \uparrow\uparrow [BC)$,. we have $\widehat{((A_1C_1), (BC))} = \widehat{((AC), (BC))} = 45^\circ$.

Sec. 19. Parallel Projecting

Let in space R there be given a plane α (the plane of projection) and a straight line l (the direction of projecting) which is not parallel to the plane α (Fig. 40). Let us then take an arbitrary point A and draw through it a line p parallel to l. Since p is not parallel to α, the line p will cut the plane α at a point A_1. The point A_1 thus obtained is called the *parallel projection of the point* A on the plane α in the direction l or simply the parallel projection: $A_1 = \mathrm{pr}_\alpha A$.

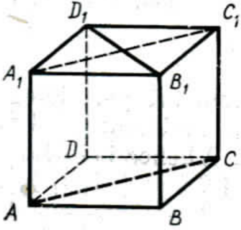


Fig. 39

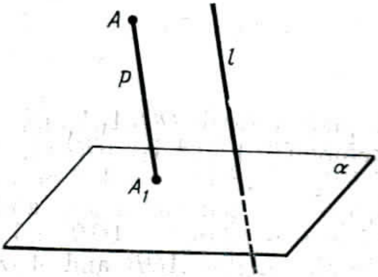


Fig. 40

If each point of a figure Φ is projected on the plane α, then the set of projections of all the points of the figure Φ form a plane figure Φ_1, which is called the *parallel projection of the figure* Φ on the plane α: $\Phi_1 = \mathrm{pr}_\alpha \Phi$.

The operation of constructing a parallel projection is called *parallel projecting*.

Let us study the elements and properties of various figures which remain unchanged in parallel projecting. In general, the properties of figures which remain unchanged in a given transformation are called *invariant* with respect to the given transformation. The numbers related with the given figure and remaining unchanged in its projection are called *invariants* of the given transformation.

For instance, any displacement in the plane keeps the distances between any two points unchanged. Consequently, distance is the invariant of this displacement.

The magnitudes of angles, as also the ratios of lengths of rectilinear segments are the invariants characterizing homothety on the plane.

Let us single out the invariant properties and the invariants of parallel projecting.

Theorem 1. *The parallel projection of a straight line p is either a point, or a straight line.*

Given: the plane of projection α, the direction of projection l, and a straight line p.

Prove: $\mathrm{pr}_\alpha\, p = p_1$, is either a straight line, or a point.

Proof. 1. If $p \parallel l$, then $\mathrm{pr}_\alpha\, p = p_1 = A_1$, where the point A_1 is the parallel projection of the line p.

2. $p \nparallel l$. Let us draw a plane β through p and parallel to l (Problem 2 of Sec. 17), which will intersect the plane α along a straight line $p_1 = \alpha \cap \beta$ (Fig. 41). Plane β is called the projecting plane.

Let us show that $p_1 = \mathrm{pr}_\alpha\, p$. Project an arbitrary point $B \in p$ on the plane α, for which purpose we draw a straight line q from point B in the plane β and parallel to l, which will intersect $p_1 = \alpha \cap \beta$ at point B_1. Obviously, $q \parallel l$. Thus, all points of the line p are projected into the line p_1, and, conversely, any point $C_1 \in p_1$ is the projection of some point $C \in p$, which was required to be proved.

Theorem 2. *Parallel projections of parallel lines (which are not parallel to the direction of projecting) are parallel.*

Given: $p \parallel q \nparallel l$, the plane of projection α.

Prove: $\mathrm{pr}_\alpha\, p \parallel \mathrm{pr}_\alpha\, q$.

Proof. We draw projecting planes β and γ through the lines p and q. In the plane β $\exists p_1 \parallel l$, in the plane γ $\exists q_1 \parallel l$ (Fig. 42). Consequently, the two intersecting lines p and p_1 in the plane β are respectively parallel to the two intersecting lines q and q_1 in the plane γ, from which it follows that $\beta \parallel \gamma$ (Theorem 3 of Sec. 17). According to Theorem 2 of Sec. 17, the plane of projection α intersects β and γ along the parallel lines $\mathrm{pr}_\alpha\, p = \alpha \cap \beta$ and $\mathrm{pr}_\alpha\, q = \alpha \cap \gamma$. Thus, $\mathrm{pr}_\alpha\, p \parallel \mathrm{pr}_\alpha\, q$, which proves the theorem.

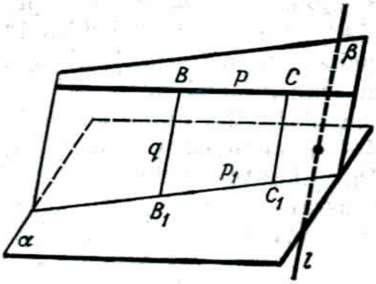


Fig. 41

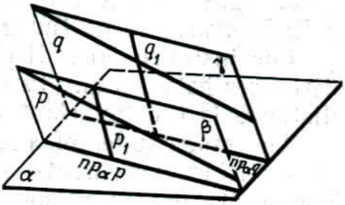


Fig. 42

Theorem 3. *The ratio of the lengths of segments of a line p is equal to the ratio of the lengths of their projections.*

Given: $[AB] \subset p$, $[BC] \subset p$, the plane of projection α, the direction of projecting l, $[A_1B_1] = \mathrm{pr}_\alpha\,[AB]$, $[B_1C_1] = \mathrm{pr}_\alpha\,[BC]$.

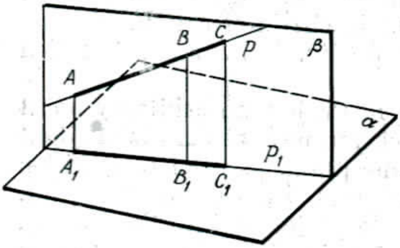


Fig. 43

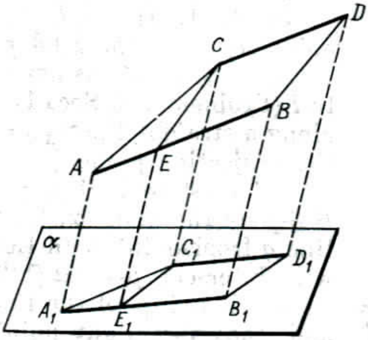


Fig. 44

Prove: $\dfrac{|AB|}{|BC|} = \dfrac{|A_1B_1|}{|B_1C_1|}$.

Proof. The line p and its projection p_1 lie in a projecting plane β. The line segments $[A_1B_1]$ and $[B_1C_1]$ are obtained as a result of intersecting the lines p and p_1 by the parallel lines (AA_1), (BB_1), and (CC_1) (see Fig. 43). Hence, $\dfrac{|AB|}{|BC|} = \dfrac{|A_1B_1|}{|B_1C_1|}$.

Problem. Prove that the ratio of the lengths of parallel line segments is equal to the ratio of the lengths of their parallel projections (Fig. 44).

Solution. Let us draw a plane β through the parallel lines (AB) and (CD). We then project the quadrilateral $ABDC$ on the plane α. As is obvious, $A_1B_1D_1C_1 = \mathrm{pr}_\alpha\, ABDC$. The line segment $[CE]$ which is parallel to $[DB]$ cuts off a parallelogram $CEBD$ from the quadrilateral $ABDC$. According to Theorem 2, the quadrilateral $C_1E_1D_1B_1$ is also a parallelogram, since $[C_1E_1] \parallel [D_1B_1]$, $[E_1B_1] \parallel [C_1D_1]$. Using Theorem 3, we now have

$$\frac{|AB|}{|EB|} = \frac{|A_1B_1|}{|E_1B_1|}, \quad \text{or} \quad \frac{|AB|}{|CD|} = \frac{|A_1B_1|}{|C_1D_1|},$$

since $|EB| = |CD|$ and $|E_1B_1| = |C_1D_1|$.

From the above theorems and the last problem it follows that parallel projecting preserves: 1°—linearity, 2°—parallelism, 3°—the ratios of the lengths of parallel segments.

Sec. 20. Representation of Figures in Solid Geometry

In *plane geometry* we are used *to represent a figure* by any figure similar to the given one. For instance, the representation of a square is a square, the representation of a regular triangle is a regular triangle, the representation of a rhombus is a rhombus with the same angles, and so on.

In solid geometry representation of figures is a more complicated task, since we come across additional difficulties. On a sheet of paper (i.e. on a plane) we have to represent both plane figures lying in different other planes, and space figures. To perform this task we use the method of parallel projecting.

The representation of a figure in solid geometry is defined as any figure which is similar to the parallel projection of the given figure on some plane (say, the plane of the drawing), provided this plane is not parallel to the direction of projecting.

Let us consider the representation of some basic figures in solid geometry.

1. *The Triangle.* According to the property 1°, the triangle is represented by a triangle. It should be borne in mind, that an equilateral triangle may be represented by a

scalene triangle if the plane containing the given triangle is not parallel to the plane of the drawing. However, the midline and the median of the triangle are represented by the midline and the median of the triangle obtained.

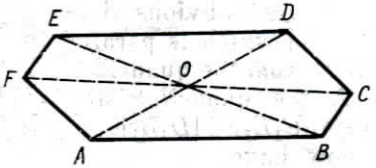


Fig. 45

2. *The Quadrilateral.* According to property 1°, the quadrilateral is represented by a quadrilateral. Due to

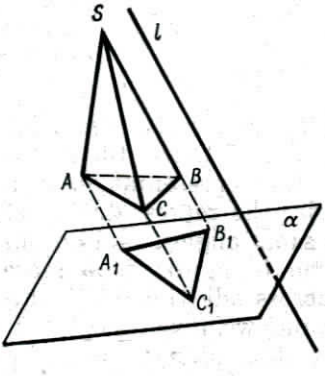
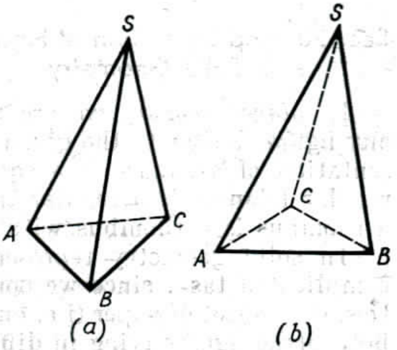


Fig. 46



(a)

(b)

Fig. 47

property 2°, the parallelogram (as well as its particular cases—the square and the rhombus) is represented by a parallelogram.

The trapezium is represented by a trapezium, the ratio of the lengths of the parallel sides of the given trapezium being equal to the ratio of the lengths of these sides in the representation.

3. *The Polygon.* The polygon is represented by a polygon with the same number of sides. In particular, a regular hexagon is represented by a hexagon, in which the diagonals

are bisected in the point of their intersection, and the opposite sides are parallel (Fig. 45).

4. *The Tetrahedron.* If the direction of projecting is parallel to a lateral edge of the tetrahedron, then the latter is projected into a triangle (Fig. 46). Of course, this is a particular case. In general, the tetrahedron is represented by a quadrilateral $SABC$, which may be convex (as in Fig. 47,*a*) or non-convex (as in Fig. 47,*b*).

5. *The Prism.* If the direction of projecting is parallel to a lateral edge of the prism, then the latter is represented

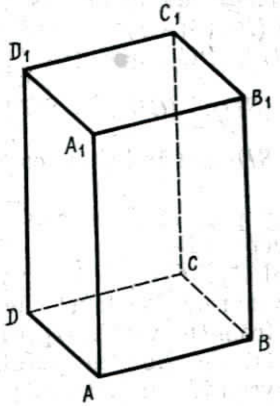


Fig. 48

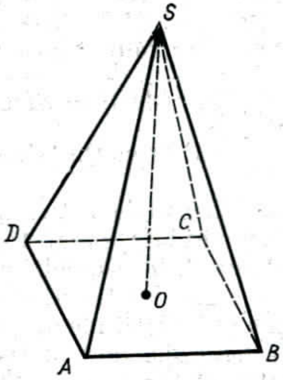


Fig. 49

by a polygon coinciding with the representation of the base of the prism. In all other cases the prism is projected into a plane figure (Fig. 48) consisting of two congruent polygons (the projections of the bases) displaced parallel with respect to each other $[(AB) \parallel (A_1B_1)$, $(BC) \parallel (B_1C_1)$, $(CD) \parallel (C_1D_1)$, $(AD) \parallel (A_1D_1)]$, and the parallelograms ABB_1A_1, BCC_1B_1, etc., which are the projections of the lateral faces of the prism.

Sec. 21. Worked Problems

Problem 1. Through the point A draw a straight line p parallel to each of the two intersecting planes α and β.

Solution. The required line p, being parallel both to α and β, must be parallel to the line of their intersection

$\alpha \cap \beta$ (by Theorem 3 of Sec. 16). Conversely, if $p \parallel \alpha \cap \beta$, then $p \parallel \alpha$ and $p \parallel \beta$ (by Theorem 1 of Sec. 16).

Thus, the problem is reduced to constructing a straight line passing through the given point A parallel to the given line $\alpha \cap \beta$ (Sec. 14).

Problem 2. The base of a regular pyramid is a square with the side a. The lateral edge of the pyramid is equal to b. Find the angles between skew edges of the pyramid (Fig. 49).

Solution. Each lateral edge of the pyramid forms skew lines only with two edges of the base, and each edge of the base, with two lateral edges. These are examples of skew lines: (SD) and (AB), (SD) and (BC), (AB) and (SD), (AB) and (SC) (Fig. 49). Since $(AB) \parallel (DC)$, and $(BC) \parallel (AD)$, then $(\widehat{(SD), (AB)}) = \widehat{SDC}$, $(\widehat{(SD), (CB)}) = \widehat{SDA}$, and $\widehat{SDC} = \widehat{SDA}$. From the triangle SDC we find that $\cos \widehat{SDC} = a/2b$.

Thus, the angle between any pair of skew lines of the given pyramid is equal to $\arccos a/2b$.

Problem 3. The quadrilateral $A_1B_1C_1D_1$ is the representation of a regular triangular pyramid. Construct the representation of its altitude dropped from the vertex D_1 (Fig. 50).

Solution. The foot of the altitude, i.e. the point O_1 is the centre of the base of the pyramid, and, consequently, this is the point of intersection of its medians. Since the medians of the triangle are depicted by the medians of its representation, the projection of the point D_1 is the point of intersection of the medians of the triangle $A_1B_1C_1$. Joining it to the vertex D_1, we get the representation of the altitude of the pyramid ($[O_1D_1]$ in Fig. 50).

Problem 4. The triangle $A_1B_1C_1$ is the representation of a triangle ABC. Construct the triangle ABC given its vertex A and a straight line $p = (ABC) \cap (A_1B_1C_1)$ (Fig. 51).

Solution. Let us consider two sides of the triangle $A_1B_1C_1$ which are not parallel to p, say $[A_1B_1]$ and $[B_1C_1]$. Obviously, the line (AB) and its representation (A_1B_1) intersect at some point $M \in p$, while the line (BC) and its representation (B_1C_1) do at some point $N \in p$. Since B_1 is the representation of B, and A_1 is the representation of A, we get $(B_1B) \parallel (A_1A)$. Consequently, the vertex B is the point of intersection of the lines (MA) and (BB_1) which are parallel to

(AA_1). Analogously, the point C is the point of intersection of the lines (NB) and (CC_1) which are parallel to (AA_1).

The above reasoning yields the following method of construction. Let us extend the sides (A_1B_1) and (B_1C_1) to intersect p at points M and N. Joining the points M and A, we draw (BB_1) parallel to (AA_1) to intersect (MA) at point B. We then join the point B thus obtained to the point N and draw (CC_1) parallel to (AA_1) to intersect (NB) at point C. ABC is the required triangle. Indeed, the vertices A, B, and C lie in one plane with the line p, B_1 and C_1 being representations of B and C $[(BB_1) \parallel (CC_1) \parallel (AA_1)]$.

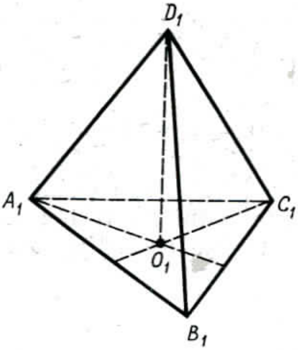


Fig. 50

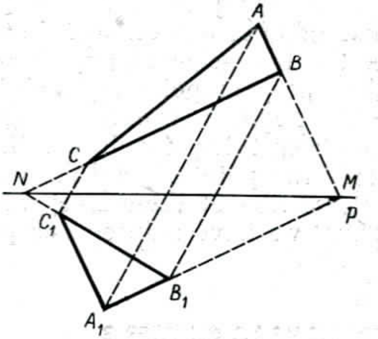


Fig. 51

Problem 5. The triangle $A_1B_1C_1$ is the representation of a triangle ABC. Construct the triangle ABC given the centre O of the circle circumscribed about the triangle, its representation O_1, and a line $p = (ABC) \cap (A_1B_1C_1)$.

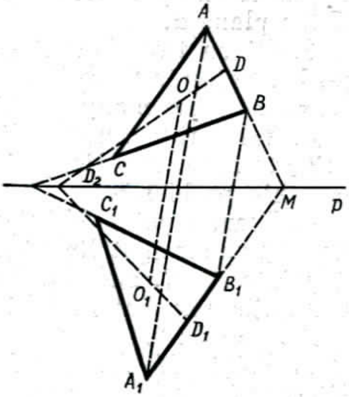


Fig. 52

Solution. Suppose (A_1B_1) is not parallel to p, and D_1 is the midpoint of $[A_1B_1]$ (Fig. 52). Obviously, $[O_1D_1]$ is the representation of the line segment $[OD]$ which is perpendicular to $[AB]$ at its midpoint D. The lines (O_1D_1)

and (OD) intersect at some point* $D_2 \subset p$. Similarly, (AB) and (A_1B_1) intersect at some point $M \in p$, (MA) being perpendicular to (OD). Consequently, the vertex A is the point of intersection of the line (A_1A) parallel to (O_1O), and the line (MA) perpendicular to (OD).

Reasoning in such a way, we come to the following method of construction. Join O_1 to D_1 which is the midpoint of the line segment $[A_1B_1]$ and extent (O_1D_1) to intersect the line p at point D_2. Joining D_2 to O, we get the line (OD). Let us now extend (A_1B_1) to intersect p at point M, and from this point draw a line perpendicular to (OD), and from the point A_1 a line parallel to (O_1O). The point of intersection of these lines will be the required vertex A. Thus, we have reduced our problem to the preceding one.

PROBLEMS TO CHAPTER 3

1. α and β are parallel planes. Prove that any straight line $p \subset \alpha$ is parallel to β.

2. Prove that the set of all straight lines passing through a point A and parallel to a given plane α is a plane parallel to the plane α.

3. Given two skew lines p and q. Through a point A not belonging to these lines draw a straight line intersecting p and q. How many solutions has this problem?

4. With the same conditions, through a point A draw a line to form skew lines with p and q. How many solutions has this problem?

5. From a point M belonging to neither of two parallel planes two straight lines are drawn intersecting the planes at points A, B, and A_1, B_1, respectively. Find the length of the line segment $[AA_1]$ if $|BB_1| = 28$ cm and $|MA| : |AB| = 5:2$.

6. Points M, N, and P are given on the edges $[DA]$, $[DB]$, and $[DC]$ of a pyramid $ABCD$. Construct the point of intersection of the plane (MNP) and the line segment $[DE]$, where E is the point of intersection of the medians of the base ABC.

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* We suppose that O_1D_1 is not parallel to p, otherwise OD is parallel to p. Consider this case independently.

7. Through the midpoints of the edges $[AD]$ and $[BD]$ of a tetrahedron a plane is drawn parallel to the edge (CD). Construct the lines along which this plane intersects the edges of the tetrahedron. Compute their lengths if $|AB| = a$, $|CD| = b$.

8. M and N are the midpoints of the edges $[AB]$ and $[AD]$ of a tetrahedron. Point P lies on the edge $[BC]$, and $|BP| : |PC| = 1 : 3$. A plane is drawn through the points M, N, and P. Construct the lines of intersection of this plane with the edges of the tetrahedron $ABCD$. Compute the lengths of these lines if ABC is a regular triangle with the side a, and $|AD| = |BD| = |CD| = b$.

9. Given a cube $ABCDA_1B_1C_1D_1$. A point M belongs to $[AA_1]$ and point N to $[BB_1]$, where $|AM| = |MA_1| = 1 : 2$ and $|BN| : |NB_1| = 2 : 1$. Construct the section figure cut off from the cube by the plane passing through the points M and N parallel to $[AC]$. Compute the area of the section if the edge of the cube is equal to a.

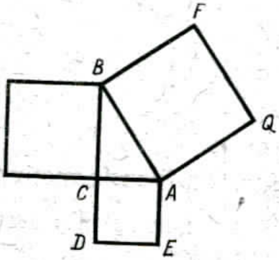


Fig. 53

10. Given a parallelepiped $ABCDA_1B_1C_1D_1$. Construct the point of intersection of the plane passing through the edges $[AB]$ and $[CD]$ with the line $(A_1C_1]$.

11. The triangle $A_1B_1C_1$ is the representation of a triangle ABC. Construct the representation of the bisector $[AD]$ if $|AB| = 10$, and $|AC| = 8$.

12. Given the representation of a triangle ABC and two of its altitudes. Construct the representation of the centre of the circle circumscribed about the triangle ABC.

13. Given the representation of a right-angled isosceles triangle. Find the representation of the centre of the incircle if the hypotenuse $|AB| = 3\sqrt{2}$.

14. Given the representation of a right-angled triangle ABC. Construct the representations of the squares $ACDE$ and $ABFQ$ (Fig. 53).

15. Given the representation of a rhombus. Construct the representation of its altitude if its acute angle is equal to 60°.

CHAPTER 4

TRANSFORMATION OF SPACE. VECTORS

Sec. 22. Transformation of Space

We shall study here the general properties of space transformation and its particular kinds—translation and homothety.

Definition 1. *Transformation of space* is defined as onto mapping such that to two different points of space there correspond two different images.

If we denote the transformation of space by f, then it follows from this definition that $f(A) \neq f(B)\ \forall A \neq B$.

Definition 2. The onto mapping preserving distances is called a *displacement*.

Consequently, we may write $\rho(f(A),\ f(B)) = \rho(A, B)\ \forall A$ and B.

Hence it follows that any displacement is a univalent mapping and therefore there exists for it an inverted mapping. A mapping which is an inverse to displacement is obviously also a displacement.

Let us mention the basic properties of displacement in space.

(1) *Identity transformation of space is a displacement.*

(2) *Displacement maps a line segment into a line segment of the same length, a straight line into a straight line, a plane into a plane.*

(3) *The displacement f maps the half-space bounded by a plane α into a half-plane bounded by the plane $f(\alpha)$.*

(4) *Displacement preserves parallelism of straight lines, planes, as well as of a straight line and a plane.*

(5) *Displacement maps an angle into an angle of the same magnitude.*

Let us prove, for example, that a displaced segment retains its length.

Suppose $A_1 = f(A)$ and $B_1 = f(B)$. For any interior point $C \in [AB]$ we have $\rho(A, C) + \rho(C, B) = \rho(A, B)$ (see Sec. 10). Then, according to the definition of displacement, for the point $C_1 = f(C)$ the equation $\rho(A_1, C_1) + \rho(C_1, B_1) = \rho(A_1, B_1)$ is satisfied, i.e. C_1 lies between A_1 and B_1, and therefore belongs to $[A_1B_1]$.

Thus, the image of any point on $[AB]$ belongs to $[A_1B_1]$. On the other hand, any point of the segment $[A_1B_1]$ is the image of some point on $[AB]$; if $M_1 \in [A_1B_1]$, then M_1 is the image of the point $M \in [AB]$ such that $|A_1M_1| = |AM|$.

We leave the proof of the other properties to the reader.

Definition 3. The figure Φ_1 is said to be *congruent* to the figure Φ if there exists a displacement which maps Φ onto Φ_1.

Congruence of figures has the following properties:

1° $\Phi \cong \Phi$ (*reflexivity*).

Indeed, an identity transformation maps Φ onto Φ.

2° If $\Phi_1 \cong \Phi$, then $\Phi \cong \Phi_1$ (*symmetry*).

In fact, if the displacement f maps Φ onto Φ_1, then the inverse transformation, which is also a displacement, maps Φ_1 onto Φ.

3°. If $\Phi \cong \Phi_1$ and $\Phi_1 \cong \Phi_2$, then $\Phi \cong \Phi_2$ (*transitivity*).

Indeed, suppose f_1 and f_2 are two displacements mapping Φ into Φ_1 and Φ_1 into Φ_2, respectively. Successive accomplishment of these displacements is, obviously, a displacement. Putting $f = f_1 \cdot f_2$, we have

$$f(\Phi) = f_2(f_1(\Phi)) = f_2(\Phi_1) = \Phi_2,$$

whence it follows that $\Phi \cong \Phi_2$.

Sec. 23. Translation in Space

Definition. *Translation* in space is defined as an onto transformation which displaces all the points in one and the same direction over one and the same distance (Fig. 54).

If f is a translation and $A_1 = f(A)$, $B_1 = f(B)$, then, according to this definition, $|AA_1| = |BB_1|$ and $[AA_1) \uparrow\uparrow [BB_1)$.

Let us show that translation in space is a displacement, i.e. that for any points A and B in space the equation $\rho(A, B) = \rho(f(A), f(B))$ is fulfilled.

To do this, we consider the quadrilateral ABB_1A_1, where $A_1 = f(A)$ and $B_1 = f(B)$ (Fig. 54). Since $|AA_1| = |BB_1|$ and $[AA_1) \uparrow\uparrow [BB_1)$, ABB_1A_1 is a parallelogram, from which it follows that $|AB| = |A_1B_1|$.

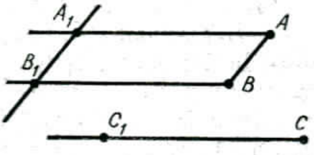


Fig. 54

Hence, in translation all the properties listed in Sec. 22 are observed.

In the next section we shall establish connection between translation and vectors in space.

Sec. 24. The Vector Defined

The notion of vector in space (in solid geometry) is introduced in the same way as in plane geometry.

Any ordered pair of noncoincident points (A, B) determines a directed line segment with the initial point A and B as the end-point (Fig. 55).

Definition 1. A *geometric vector*, or simply a *vector* $\overrightarrow{AB}$ is described as a directed line segment with A as the initial point and B as the terminal point.

The end of a vector is marked with an arrow. It is common practice to denote a vector by a single bold symbol (usually the lower-case Latin letters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc.), or in the following way: $\vec{a}$, $\vec{b}$, $\vec{c}$, ...

The vector whose initial point coincides with its terminal point is called a *null* vector. The *length* of the vector $\overrightarrow{AB}$ is equal to the length of the line segment $[AB]$. The length of a null vector is obviously equal to zero. The length of a vector is denoted as follows: $|\overrightarrow{AB}|$, or $|\vec{a}|$.

Direction of the vector $\overrightarrow{AB}$ is defined as the direction determined by the ray $[AB)$. This means that direction in space may be specified by a vector, or more exactly, by its direction.

Using the notion of vector, we may define translation in space as a transformation which displaces all points of a space in the direction of a given vector $\overrightarrow{AB}$ over one and the same distance equal to the length of the vector $\overrightarrow{AB}$ (Fig. 55).

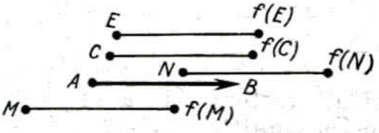


Fig. 55

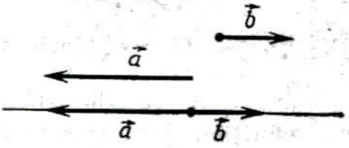


Fig. 56

Thus, any two vectors having equal lengths and the same direction specify one and the same translation.

Definition 2. Two vectors are said to be *equal* if they specify one and the same translation in space. Or, in other words, vectors are said to be equal if their lengths are equal and the directions coincide.

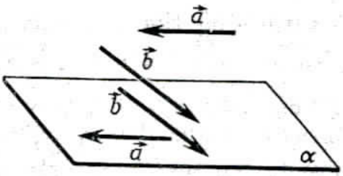


Fig. 57

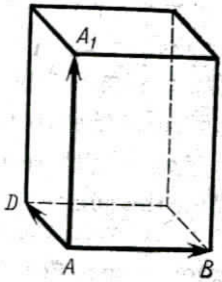


Fig. 58

It follows from this definition that:

(1) the initial point of a vector may be transferred to any point of space, therefore a geometric vector is called a *free* vector;

(2) two vectors lying on parallel lines may be transferred on one line (Fig. 56);

(3) any pair of vectors may be transferred to one plane (Fig. 57).

Definition. Vectors lying on parallel straight lines (or on one and the same straight line) are termed *collinear*. Vectors parallel to one and the same plane are called *coplanar*.

Consequently, any pair of vectors are coplanar. Three vectors can be non-coplanar. For instance, the three vectors $\overrightarrow{AB}$, $\overrightarrow{AA_1}$, and $\overrightarrow{AD}$ coinciding with the edges of a rectangular parallelepiped (Fig. 58) are not parallel to one plane, i.e. they are non-coplanar.

Sec. 25. The Sum of Vectors

Let us consider the set $\mathfrak{M}$ of all vectors $\vec{a}$ in a space R.

Definition 1. The operation of addition is defined for the set $\mathfrak{M}$, if any pair of vectors $\vec{a}$ and $\vec{b}$ is related with the vector $\vec{c} \subset \mathfrak{M}$ called the *sum* of $\vec{a}$ and $\vec{b}$: $\vec{c} = \vec{a} + \vec{b}$. The following conditions (the *axioms of addition*) must be fulfilled for any $\vec{a}$, $\vec{b}$, $\vec{c}$:

1°. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

2°. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

3°. $\vec{a} + \vec{0} = \vec{a}$.

4°. $\forall \vec{a} \subset \mathfrak{M} \exists\, \vec{b} \in \mathfrak{M}$ so that $\vec{a} + \vec{b} = \vec{0}$, the vector $\vec{b}$ being unique. The vector $\vec{b}$ satisfying the last equation is called *opposite* with respect to $\vec{a}$, and is denoted $-\vec{a}$.

To find the sum of two given vectors $\vec{a}$ and $\vec{b}$ we use the connection between the vectors and translation (see Sec. 24). The vector $\vec{a}$ specifies a translation which transforms the point A into B so that $\overrightarrow{AB} = \vec{a}$. The vector $\vec{b}$ also specifies a translation which transforms the point B into the point C so that $\overrightarrow{BC} = \vec{b}$ (Fig. 59).

Consequently, as a result of two successive translations defined first by the vector $\vec{a}$ and then by the vector $\vec{b}$ the point A is mapped into the point C, i.e. the initial point of the first vector $\overrightarrow{AB}$ is brought to the terminal point of the second vector $\overrightarrow{BC}$.

Definition 2. Successive performance of two transformations is called the *composition* of these transformations.

Let us show that a composition of two translations is also a translation defined by the vector $\overrightarrow{AC}$.

Let A_1 be an arbitrary point in space (Fig. 59). As a result of the first translation onto the vector $\vec{a}$ it is transferred into the point B_1 so that $\overrightarrow{A_1B_1} = \overrightarrow{AB}$. During the second translation onto the vector $\vec{b}$ the point B_1 is brought to C_1, and $\overrightarrow{B_1C_1} = \overrightarrow{BC}$. Let us prove that $\overrightarrow{A_1C_1} = \overrightarrow{AC}$.

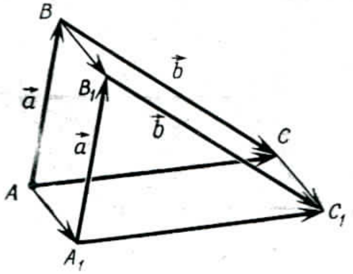


Fig. 59

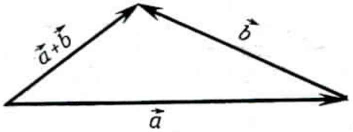


Fig. 60

Since AA_1B_1B is a parallelogram ($\overrightarrow{AB} = \overrightarrow{A_1B_1}$), we have $\overrightarrow{AA_1} = \overrightarrow{BB_1}$; BB_1C_1C is also a parallelogram ($\overrightarrow{BC} = \overrightarrow{B_1C_1}$), hence, $\overrightarrow{BB_1} = \overrightarrow{CC_1}$. Whence, by virtue of the transitive property, it follows that $\overrightarrow{AA_1} = \overrightarrow{CC_1}$, and hence, AA_1C_1C is a parallelogram, i.e. $\overrightarrow{A_1C_1} = \overrightarrow{AC}$.

Thus, a *composition of two translations specified by two vectors* $\vec{a}$ *and* $\vec{b}$ *is a translation specified by the vector* $\vec{c}$ *joining the initial point of the first* ($\vec{a}$) *to the terminal point of the second* ($\vec{b}$).

With this reasoning we arrive to the following definition.

Definition 3. The *sum* of two vectors $\vec{a}$ and $\vec{b}$ is defined as the vector joining the initial point of the first vector to the terminal point of the second, provided the initial point of the vector $\vec{b}$ and the terminal point of the vector $\vec{a}$ are brought together (Fig. 60).

If the vectors $\vec{a}$ and $\vec{b}$ are collinear, then the vector $\vec{a} + \vec{b} = \vec{c}$ lies on one line with them (Fig. 61). If $\vec{a}$ and $\vec{b}$ are non-collinear, then the vectors $\vec{a}$, $\vec{b}$, and $\vec{c} = \vec{a} + \vec{b}$

form a triangle (Fig. 62). That is why the rule for adding the vectors mentioned in the definition is called the "triangle rule".

Let us show that this rule satisfies all the conditions of 1° through 4°.

1°. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

From Fig. 62 it is seen that $\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, $\vec{b} + \vec{a} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$, i.e. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

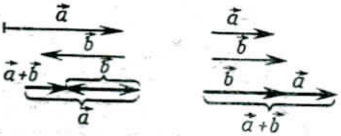


Fig. 61

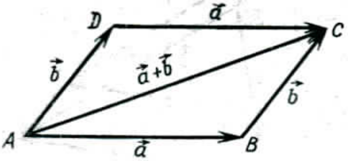


Fig. 62

In proving this property we get another rule of vectors adding called the "parallelogram rule".

If the vectors $\vec{a}$ and $\vec{b}$ are reduced to a common origin, then their sum is a vector coinciding with the diagonal of a parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ as sides, and emanating from their common origin (Fig. 62).

2°. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Let us arrange the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ so that the initial point of the next vector coincides with the terminus of the preceding one. Figure 63,*a* illustrates the sequence of operations: first $\vec{a} + \vec{b} = \overrightarrow{AC}$, then $\overrightarrow{AC} + \vec{c} = \overrightarrow{AD}$. In Fig. 63,*b* we first get the vector $\vec{b} + \vec{c} = \overrightarrow{BD}$, and then $\vec{a} + \overrightarrow{BD} = \overrightarrow{AD}$. We see that the results have coincided, i.e. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

The equation thus obtained enables us to write each sum without using parentheses.

3°. $\vec{a} + \vec{0} = \vec{a}$, which is obvious.

4°. $\forall \vec{a} \exists$ the unique vector $\vec{b}$ which is equal to $\vec{a}$ in magnitude (i.e. $\vec{b}$ and $\vec{a}$ have equal lengths) and is oppositely directed.

For this vector $\vec{b}$, as is obvious, $\vec{a} + \vec{b} = \vec{0}$. Such vector $\vec{b}$ is said to be opposite to the vector $\vec{a}$. According to the notation introduced: $\vec{b} = -\vec{a}$.

Note. While considering property 2° we got the sum of three vectors. Using the same method, we can obtain the

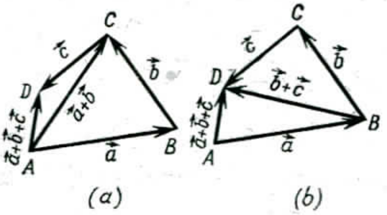


Fig. 63

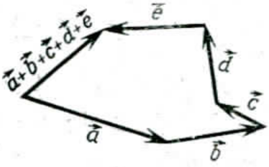


Fig. 64

sum of any finite number of vectors:

$$\vec{a}+\vec{b}+\vec{c}+\vec{d}=(\vec{a}+\vec{b}+\vec{c})+\vec{d},$$

$$\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}=(\vec{a}+\vec{b}+\vec{c}+\vec{d})+\vec{e}.$$

This sum can be constructed by the following rule.

If the initial point of the next summand is brought to the terminus of each addend vector. then the, vector joining the initial point of the first vector to the terminal point of the last one will be the sum of all these summands (Fig. 64).

Sec. 26. Subtraction of Vectors. Multiplication of a Vector by a Number

Subtraction of vectors is the inverse operation of addition.

Definition 1. The *difference* between two vectors $\vec{a}$ and $\vec{b}$ is defined as a new vector $\vec{c}$ which together with the vector $\vec{b}$ yields the vector $\vec{a}$:

$$\vec{c}=\vec{a}-\vec{b}, \quad \text{if} \quad \vec{c}+\vec{b}=\vec{a}.$$

Let us show that for any two vectors $\vec{a}$ and $\vec{b}$ there exists the difference $\vec{c} = \vec{a} - \vec{b}$ which is unique.

Indeed, $\forall \vec{b} \exists -\vec{b}$ so that $\vec{b} + (-\vec{b}) = \vec{0}$. Suppose that $\vec{a} - \vec{b} = \vec{c}$ and $\vec{a} - \vec{b} = \vec{d}$. According to the definition, this means that $\vec{c} + \vec{b} = \vec{a}$ and $\vec{d} + \vec{b} = \vec{a}$. Adding the vector $-\vec{b}$ (which exists by virtue of property 4°), we get

$$\vec{c}+\vec{b}+(-\vec{b})=\vec{a}+(-\vec{b}) \text{ and } \vec{d}+\vec{b}+(-\vec{b})=\vec{a}+(-\vec{b})$$

or

$$\vec{c}=\vec{a}+(-\vec{b}) \text{ and } \vec{d}=\vec{a}+(-\vec{b}),$$

whence it follows that $\vec{c} = \vec{d} = \vec{a} + (-\vec{b})$, i.e. the difference between the two vectors $\vec{a}$ and $\vec{b}$ is equal to the sum of the vectors $\vec{a}$ and $-\vec{b}$:

$$\vec{a}-\vec{b}=\vec{a}+(-\vec{b}).$$

From this formula and the triangle rule follows the method of construction of the difference between two vectors: if the vectors $\vec{a}$ and $\vec{b}$ are reduced to a common origin,

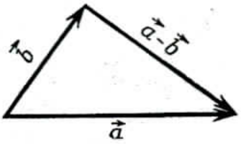


Fig. 65

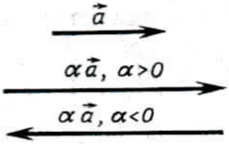


Fig. 66

then the difference $\vec{a} - \vec{b}$ is the vector joining the terminus of the subtrahend $\vec{b}$ to the terminus of the minuend $\vec{a}$ (Fig. 65).

Definition 2. *The product of a vector* $\vec{a}$ *by a number* α is defined as a new vector $\alpha\vec{a}$, whose length is equal to $|\alpha|\,|\vec{a}|$, the direction coinciding with the direction of the vector $\vec{a}$ if $\alpha > 0$, or being in the opposite sense if $\alpha < 0$ (Fig. 66). If $\alpha = 0$, the product is the null vector, i.e. $\alpha\vec{a} = \vec{0}$.

The operation of multiplication of a vector by a number possesses the following properties.

1°. $\alpha(\beta\vec{a}) = (\alpha\beta)\vec{a}$ — *associative property with respect to the numerical factors.*

Proof. 1. According to Definition 2 and the properties of the absolute value, we have

$$|\alpha(\beta\vec{a})| = |\alpha| \cdot |\beta\vec{a}| = |\alpha| \cdot |\beta| \cdot |\vec{a}| = |\alpha\beta| \cdot |\vec{a}|,$$

i.e. $|\alpha(\beta\vec{a})| = |(\alpha\beta)\vec{a}|$.

2. If α and β have the same signs then $\alpha(\beta\vec{a}) \uparrow\uparrow \vec{a}$ and $(\alpha\beta)\vec{a} \uparrow\uparrow \vec{a}$, i.e. $\alpha(\beta\vec{a}) \uparrow\uparrow (\alpha\beta)\vec{a}$. If α and β have opposite signs, then $\alpha(\beta\vec{a}) \uparrow\downarrow \vec{a}$ and $(\alpha\beta)\vec{a} \uparrow\downarrow \vec{a}$. Thus, in this case also $\alpha(\beta\vec{a}) \uparrow\downarrow (\alpha\beta)\vec{a}$.

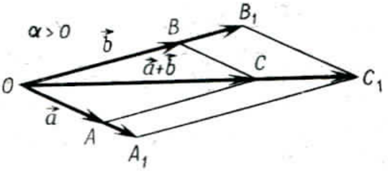


Fig. 67

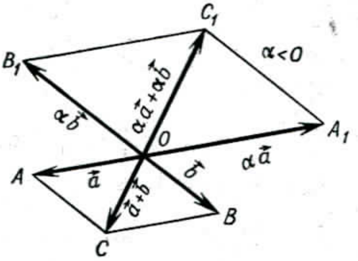


Fig. 68

From the above proof it follows that $\alpha(\beta\vec{a}) = (\alpha\beta)\vec{a}$.

2°. $\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}$ —*distributive property with respect to the sum of vectors.*

Proof. Let us draw from the common origin O the vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ and construct their sum $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. If $\alpha > 0$, then $\alpha\vec{a} = \overrightarrow{OA_1} \uparrow\uparrow \vec{a}$, $\alpha\vec{b} = \overrightarrow{OB_1} \uparrow\uparrow \vec{b}$, i.e. $\alpha\vec{a} + \beta\vec{b} = \overrightarrow{OC_1}$ (Fig. 67). Since $(A_1C_1) \parallel (OB)$ and $(AC) \parallel (OB)$, then $(A_1C_1) \parallel (AC)$, and, hence, $\angle OAC \cong \angle OA_1C_1$. Besides, $|OA| : |OA_1| = |AC| : |A_1C_1| = \alpha$. Therefore, the triangle OAC is similar to the triangle OA_1C_1, from which it follows that $[OC]$ and $[OC_1]$ lie on one line; hence, $|OC_1| = \alpha |OC|$, i.e. $\overrightarrow{OC_1} = \alpha\overrightarrow{OC} = \alpha(\vec{a} + \vec{b})$, or $\alpha\vec{a} + \alpha\vec{b} = \alpha(\vec{a} + \vec{b})$.

The case $\alpha < 0$ is proved in a similar way (Fig. 68).

3°. $(\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$—*distributive property with respect to the sum of numbers.*

Proof. 1. If α and β have the same sign, then the vectors $\alpha\vec{a}$, $\beta\vec{a}$, and $(\alpha + \beta)\vec{a}$ are in the same direction. In this case we have $\alpha\vec{a} + \beta\vec{a} \uparrow\uparrow (\alpha + \beta)\vec{a}$ and $|\alpha\vec{a} + \beta\vec{a}| = |\alpha\vec{a}| + |\beta\vec{a}| = |(\alpha + \beta)\vec{a}|$.

Thus, the vectors $\alpha\vec{a} + \beta\vec{a}$ and $(\alpha + \beta)\vec{a}$ are in the same direction and have equal lengths, hence they are equal.

2. If α and β have opposite signs, for instance, $\alpha < 0$, $\beta > 0$, and $|\alpha| < |\beta|$, then $\alpha + \beta > 0$, $|\alpha + \beta| = |\beta| - |\alpha|$. In this case $\alpha\vec{a} \uparrow\downarrow \vec{a}$, $\beta\vec{a} \uparrow\uparrow \vec{a}$, $(\alpha + \beta)\vec{a} \uparrow\uparrow \vec{a}$. According to the rule for adding oppositely directed vectors, we get

$$\alpha\vec{a} + \beta\vec{a} \uparrow\uparrow (\alpha + \beta)\vec{a}, \qquad |\alpha\vec{a} + \beta\vec{a}| = |\beta\vec{a}| - |\alpha\vec{a}| = |(\alpha + \beta)\vec{a}|.$$

Thus, in this case we also have $\alpha\vec{a} + \beta\vec{a} = (\alpha + \beta)\vec{a}$.

Sec. 27. A Linear Combination of Vectors. The Conditions of Collinearity and Coplanarity

The expression $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$, in which at least one of the numbers α, β, γ is not equal to zero, is called the linear combination of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. The equality $\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ means that the vector $\vec{d}$ is a linear combination of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, i.e. it is linearly expressed in terms of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. The numbers α, β, γ are called *decomposition coefficients* of the vector $\vec{d}$ with respect to the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Let us find out for which conditions a vector is a linear combination of other vectors.

Theorem 1. *Two non-zero vectors $\vec{a}$ and $\vec{b}$ are related as $\vec{b} = \alpha\vec{a}$ if and only if they are collinear.*

Proof. 1. Let $\vec{b} = \alpha\vec{a}$. According to Definition 2 of Sec. 26, this means that either $\vec{a} \uparrow\uparrow \vec{b}$ $(\alpha > 0)$, or $\vec{a} \uparrow\downarrow \vec{b}$ $(\alpha < 0)$, i.e. $\vec{a}$ and $\vec{b}$ are collinear.

2. Let $\vec{a}$ and $\vec{b}$ be collinear and $x = |\vec{b}| : |\vec{a}|$. If $\vec{b} \uparrow\uparrow \vec{a}$, then $\vec{b} = x\vec{a}$; and if $\vec{b} \uparrow\downarrow \vec{a}$, then $\vec{b} = -x\vec{a}$.

Thus, $\vec{b} = \alpha\vec{a}$, where $\alpha = x$, if $\vec{a} \uparrow\uparrow \vec{b}$, and $\alpha = -x$, if $\vec{a} \uparrow\downarrow \vec{b}$.

Theorem 2. *One of the three non-zero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is expressed in terms of a linear combination of the remaining vectors if and only if they are coplanar.*

Proof. 1. Suppose $\vec{c} = \alpha\vec{a} + \beta\vec{b}$. If the vectors $\vec{a}$ and $\vec{b}$ are collinear, then the vector $\vec{c}$ is also collinear with them. But if $\vec{a}$ and $\vec{b}$ are not collinear, then $\vec{c}$ is the diagonal of a parallelogram constructed on the vectors $\alpha\vec{a}$ and $\beta\vec{b}$, and, consequently, it lies in one plane with $\vec{a}$ and $\vec{b}$.

2. Suppose the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar. If there is at least one pair of collinear vectors, for instance, $\vec{a}$ and $\vec{b}$, then, by Theorem 1, $\vec{b} = \alpha\vec{a}$ and we may write $\vec{b} = \alpha\vec{a} + 0\cdot\vec{c}$.

Let us now suppose that there are no collinear vectors among $\vec{a}$, $\vec{b}$, and $\vec{c}$. From their common origin O (Fig. 69) and from the terminal point of the vector $\vec{c}$ we draw straight lines parallel to $\vec{a}$ and $\vec{b}$. As is obvious from Fig. 69, $\vec{c} = \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. Since $\overrightarrow{OA}$ is collinear with $\vec{a}$, and $\overrightarrow{OB}$ is collinear with $\vec{b}$, we have $\overrightarrow{OA} = \alpha\vec{a}$, $\overrightarrow{OB} = \beta\vec{b}$ and, hence, $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

Let us now show that in the expression $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ the numbers α and β are defined uniquely ($\vec{a}$ and $\vec{b}$ are not collinear). Suppose that there exist two decompositions $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ and $\vec{c} = \alpha_1\vec{a} + \beta_1\vec{b}$. Substracting the second equation from the first one, we get $(\alpha_1 - \alpha)\,\vec{a} = (\beta - \beta_1)\,\vec{b}$. Since the vectors $\vec{a}$ and $\vec{b}$ are not collinear (by hypothesis), it follows from the last equation that $\alpha = \alpha_1$ and $\beta = \beta_1$.

Theorem 3. *Any vector $\vec{d}$ is a linear combination of three non-coplanar vectors $\vec{a}$, $\vec{b}$,* and *$\vec{c}$, the numbers α, β, and γ*

in the decomposition $\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ *being defined uniquely.*

Proof. Let us draw the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ from the common origin O. If $\vec{d}$ is coplanar with any pair of vectors, say, $\vec{a}$ and $\vec{b}$, then, according to Theorem 2, $\vec{d} = \alpha\vec{a} + \beta\vec{b}$ and in this case the equation $\vec{d} = \alpha\vec{a} + \beta\vec{b} + 0\cdot\vec{c}$ $(\gamma = 0)$ is valid.

Let us suppose now that $\vec{d}$ is not coplanar with any pair of the given vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. From the terminal point of the vector $\vec{d} = \overrightarrow{OD}$ we draw a straight line parallel to the vector $\vec{c}$ (Fig. 70), which will cut the plane OAB ($\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$) at point D_1. From this point we draw $[D_1A_1] \parallel [OB]$ and $[D_1B_1] \parallel [OA]$. According to Theorems 2 and 1

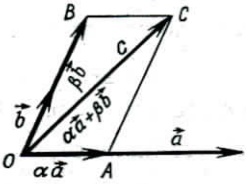


Fig. 69

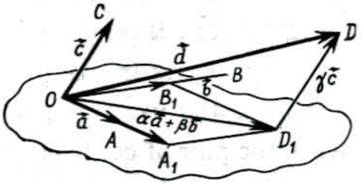


Fig. 70

$$\overrightarrow{OD_1} = \overrightarrow{OA_1} + \overrightarrow{OB_1} = \alpha\vec{a} + \beta\vec{b},$$

$$\vec{d} = \overrightarrow{OD} = \overrightarrow{OD_1} + \overrightarrow{D_1D} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}.$$

The uniqueness of the decomposition obtained is proved in the same way as in Theorem 2.

Example 1. Given in a triangle ABC are the vectors $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{BM} = \vec{d}$, where $[BM]$ is a median of the triangle ABC. Express the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of the vectors $\vec{a}$ and $\vec{d}$.

Solution.

(1) $\overrightarrow{MC} = \overrightarrow{BC} - \overrightarrow{BM} = \vec{a} - \vec{d}$;

(2) $\overrightarrow{AC} = 2\overrightarrow{MC} = 2\vec{a} - 2\vec{d}$;

(3) $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = 2\vec{a} - 2\vec{d} - \vec{a} = \vec{a} - 2\vec{d}$.

Example 2. In a regular triangular pyramid there given three vectors: $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{SB} = \vec{b}$, and $\overrightarrow{SO} = \vec{d}$, where $[SO]$ is the altitude of the pyramid. Express the vectors coinciding with the remaining edges of the pyramid in terms of the vectors $\vec{a}$, $\vec{b}$, $\vec{d}$.

Solution.

(1) $\overrightarrow{OB} = \overrightarrow{SB} - \overrightarrow{SO} = \vec{b} - \vec{d}$;

(2) $\overrightarrow{BD} = \frac{3}{2}\overrightarrow{BO} = \frac{3}{2}(\vec{d} - \vec{b})$; D is the midpoint of $[AC]$;

(3) $\overrightarrow{DC} = \overrightarrow{BC} - \overrightarrow{BD} = \vec{a} - \frac{3}{2}\vec{d} + \frac{3}{2}\vec{b}$;

(4) $\overrightarrow{AC} = 2\overrightarrow{DC} = 2\vec{a} - 3\vec{d} + 3\vec{b}$;

(5) $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = -\vec{a} + 2\vec{a} - 3\vec{d} + 3\vec{b} =$
$= -3\vec{d} + \vec{a} + 3\vec{b}$;

(6) $\overrightarrow{SC} = \overrightarrow{SB} + \overrightarrow{BC} = \vec{b} + \vec{a}$;

(7) $\overrightarrow{SA} = \overrightarrow{SB} - \overrightarrow{AB} = \vec{b} - \vec{a} + 3\vec{d} - 3\vec{b} = 3\vec{d} -$
$- 2\vec{b} - \vec{a}$.

Sec. 28. Scalar Product of Vectors

Problem. A body is rolling down an inclined plane by gravity. Find the work performed by this force as the body moves from the point A to the base of the inclined plane if the mass of the body is equal to m, and the plane is inclined at an angle φ.

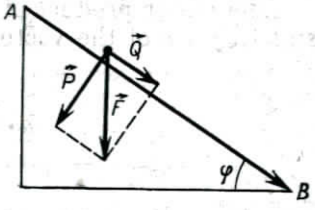


Fig. 71

Solution. The body is acted upon by a vertical force $\vec{F}$, where $|\vec{F}| = mg$. Decomposing $\vec{F}$ into two components, we get $\vec{F} = \vec{P} + \vec{Q}$ (Fig. 71), where the force $\vec{P}$ is perpendicular to the direction of motion and, consequently, does not perform work, while the force $\vec{Q}$ acts in the direction of the rectilinear displacement AB and performs work which is equal to $|\vec{Q}| \cdot |\overrightarrow{AB}|$. Taking into account that $|\vec{Q}| = |\vec{F}| \times \cos\varphi$, we finally find $W = |\vec{F}| \cdot |\overrightarrow{AB}| \cdot \cos\varphi$. The scalar quantity (work) is obtained as a result of a new operation on vectors: the force vector $\vec{F}$ and the displacement vector $\overrightarrow{AB}$.

Let us introduce this operation for arbitrary vectors $\vec{a}$ and $\vec{b}$.

Definition. The *scalar product* of two non-zero vectors $\vec{a}$ and $\vec{b}$ is a number equal to the product of their lengths by the cosine of the angle between them.

If one of the vectors is equal to zero (i.e. a null vector) then the angle between them is indeterminate, and the above definition has no sense. In this case we suppose that the scalar product is equal to zero.

The scalar product is denoted in the following way: $\vec{a}\vec{b}$, or $\vec{a} \cdot \vec{b}$, or $(\vec{a}, \vec{b})$.

If $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ then the equality $\vec{a}\vec{b} = 0$ is possible if and only if the vectors $\vec{a}$ and $\vec{b}$ form a right angle.

Indeed, the equation $|\vec{a}| \cdot |\vec{b}| \times \cos\varphi = 0$, where $|\vec{a}| \neq 0$, $|\vec{b}| \neq 0$, is possible if and only if $\cos\varphi = 0$ $(0° \leqslant \varphi \leqslant 180°)$, i.e. if $\varphi = 90°$.

The scalar product $\vec{a} \cdot \vec{b} < 0$, if $90° < \varphi \leqslant 180°$ (the vectors form an obtuse angle), and $\vec{a} \cdot \vec{b} > 0$, if $0° \leqslant \varphi < 90°$ (the vectors form an acute angle).

The scalar product of the vector $\vec{a}$ by itself is termed the *scalar square* of the vector:

$$\vec{a} \cdot \vec{a} = \vec{a}^2 = |\vec{a}|^2,$$

i.e. the scalar square of a vector is equal to the square of its length.

Problem. Prove the equation $(\vec{a} + \vec{b})^2 = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$.

Solution. Suppose $\vec{a}$ and $\vec{b}$ are not collinear. Then the vectors $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$ form a triangle. By the law of cosines we have

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 - 2|\vec{a}|\cdot|\vec{b}|\cos\widehat{ABC} + |b|^2 =$$

$$= |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos(\widehat{\vec{a},\vec{b}}) + |\vec{b}|^2.$$

Hence, $(\vec{a}+\vec{b})^2 = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$.

We leave the proof of this formula for collinear vectors $\vec{a}$ and $\vec{b}$ to the reader.

Sec. 29. Arithmetic Properties of a Scalar Product

A scalar product of vectors has the following properties:

1°. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (*commutative property*).

Proof. The angle between vectors is understood as the angle between their directions. Consequently, $(\widehat{\vec{a},\ \vec{b}}) = (\widehat{\vec{b},\ \vec{a}})$ and

$$\vec{b}\cdot\vec{a} = |\vec{b}|\cdot|\vec{a}|\cdot\cos(\widehat{\vec{b},\ \vec{a}}) = |\vec{a}|\cdot|\vec{b}|\cdot\cos(\widehat{\vec{a},\vec{b}}) = \vec{a}\cdot\vec{b}.$$

2°. $(\alpha\vec{a})\cdot\vec{b} = \alpha(\vec{a}\cdot\vec{b})$ (*associative property with respect to the numerical factors*).

Proof. For $\alpha = 0$ the equation is obvious. For $\alpha \neq 0$ we have

$$(\alpha\vec{a})\cdot\vec{b} = |\alpha\vec{a}|\cdot|\vec{b}|\cdot\cos(\widehat{\alpha\vec{a},\vec{b}}) = |\alpha|\cdot|\vec{a}|\cdot|\vec{b}|\cdot\cos(\widehat{\alpha\vec{a},\vec{b}}).$$

If $\alpha > 0$, then $|\alpha| = \alpha$, $\alpha\vec{a} \uparrow\uparrow \vec{a}$, $\cos(\widehat{\alpha\vec{a},\ \vec{b}}) = \cos(\widehat{\vec{a},\vec{b}})$. Therefore

$$((\alpha\vec{a})\cdot\vec{b}) = \alpha|\vec{a}|\cdot|\vec{b}|\cdot\cos(\widehat{\vec{a},\vec{b}}) = \alpha(\vec{a}\cdot\vec{b}).$$

If $\alpha < 0$, then $|\alpha| = -\alpha$, $\cos(\widehat{\alpha\vec{a},\ \vec{b}}) = \cos(\widehat{-\vec{a},\ \vec{b}}) = \cos[\pi - (\widehat{\vec{a},\ \vec{b}})] = -\cos(\widehat{\vec{a},\ \vec{b}})$. Therefore

$$((\alpha\vec{a})\cdot\vec{b}) = -\alpha|\vec{a}|\cdot|\vec{b}|\cdot(-\cos(\widehat{\vec{a},\vec{b}})) = \alpha(\vec{a}\cdot\vec{b}).$$

Problem. Prove that $(\vec{a}-\vec{b})^2=\vec{a}^2-2\vec{a}\cdot\vec{b}+\vec{b}^2$.

Solution. Using the result of the problem solved in Sec. 28 and Property 2°, we have

$$(\vec{a}+(-\vec{b}))^2=\vec{a}^2+2\vec{a}\cdot(-\vec{b})+(-\vec{b})^2=\vec{a}^2-2\vec{a}\cdot\vec{b}+\vec{b}^2.$$

3°. $(\vec{a}+\vec{b})\,\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$ *(distributive property with respect to the sum of vectors).*

Proof. Using the formulas

$$(\vec{p}\pm\vec{d})^2=\vec{p}^2\pm2\vec{p}\vec{d}+\vec{d}^2,$$

for the vectors $\vec{p}=\frac{\vec{a}}{2}+\vec{b}$ and $\vec{d}=\frac{\vec{a}}{2}+\vec{c}$, we have

$$\left(\left(\frac{\vec{a}}{2}+\vec{b}\right)+\left(\frac{\vec{a}}{2}+\vec{c}\right)\right)^2=$$

$$=\left(\frac{\vec{a}}{2}+\vec{b}\right)^2+2\left(\frac{\vec{a}}{2}+\vec{b}\right)\left(\frac{\vec{a}}{2}+\vec{c}\right)+\left(\frac{\vec{a}}{2}+\vec{c}\right)^2$$

and

$$\left(\left(\frac{\vec{a}}{2}+\vec{b}\right)-\left(\frac{\vec{a}}{2}+\vec{c}\right)\right)^2=$$

$$=\left(\frac{\vec{a}}{2}+\vec{b}\right)^2-2\left(\frac{\vec{a}}{2}+\vec{b}\right)\left(\frac{\vec{a}}{2}+\vec{c}\right)+\left(\frac{\vec{a}}{2}+\vec{c}\right)^2.$$

Adding these equations, we get

$$(\vec{a}+(\vec{b}+\vec{c}))^2+(\vec{b}-\vec{c})^2=2\left(\frac{\vec{a}}{2}+\vec{b}\right)^2+2\left(\frac{\vec{a}}{2}+\vec{c}\right)^2.$$

Using the formula for the square of a sum once again, we simplify both members of the last equation:

$$\vec{a}^2+2\vec{a}\,(\vec{b}+\vec{c})+(\vec{b}+\vec{c})^2+\vec{b}^2-2\vec{b}\cdot\vec{c}+\vec{c}^2=$$

$$=\vec{a}^2+2\vec{a}\cdot\vec{b}+2\vec{b}^2+2\vec{a}\cdot\vec{c}+2\vec{c}^2;$$

$$\vec{a}^2+2\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}^2+2\vec{b}\cdot\vec{c}+\vec{c}^2+\vec{b}^2-2\vec{b}\cdot\vec{c}+\vec{c}^2=$$

$$=\vec{a}^2+2\vec{a}\cdot\vec{b}+2\vec{b}^2+2\vec{a}\cdot\vec{c}+2\vec{c}^2.$$

Collecting similar terms and reducing, we get

$$\vec{a}\,(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c},$$

which was required to be proved.

Properties 1° to 3° yield the following rule for a scalar multiplication of linear combinations of vectors:

$$(\alpha\vec{a}+\beta\vec{b})\cdot(\gamma\vec{c}+\delta\vec{d})=\alpha\gamma\vec{a}\cdot\vec{c}+\beta\gamma\cdot\vec{b}\cdot\vec{c}+\alpha\delta\vec{a}\cdot\vec{d}+\beta\delta\vec{b}\cdot\vec{d},$$

consisting in that linear combinations of vectors are multiplied according to the standard rule for multiplying a polynomial by a polynomial.

Example. The sides of a parallelogram are: $|AB| = \sqrt{3}$, $|AD| = 1$; the angle between them is equal to 30°. Find the angles between the diagonals of the parallelogram.

Solution. Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$. Then $\overrightarrow{AC} = \vec{a} + \vec{b}$, $\overrightarrow{DB} = \vec{a} - \overleftarrow{b}$'

$$\cos\varphi=\frac{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}|\cdot|\vec{a}-\vec{b}|}.$$

Since $(\vec{a}+\vec{b})^2 = |\vec{a}+\vec{b}|^2$ and $|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})^2$, we have

$$|\vec{a}+\vec{b}|\cdot|\vec{a}-\vec{b}|=\sqrt{(\vec{a}+\vec{b})^2\cdot(\vec{a}-\vec{b})^2}.$$

Therefore

$$\cos\varphi=\frac{|\vec{a}|^2-|\vec{b}|^2}{\sqrt{(\vec{a}^2+2\vec{a}\vec{b}+\vec{b}^2)(\vec{a}^2-2\vec{a}\vec{b}+\vec{b}^2)}}=$$

$$=\frac{3-1}{\sqrt{(3+2\sqrt{3}\cdot\sqrt{3}/2+1)(3-2\sqrt{3}\cdot\sqrt{3}/2+1)}}.$$

Hence, $\cos\varphi = 2/\sqrt{7}$ and $\varphi = \arccos(2/\sqrt{7})$.

Sec. 30. Vector Product of Vectors

Problem. The arms of scales have different lengths: $|OA| = a$ and $|OB| = b$. A force $\vec{F}$ is applied at the point B, forming an angle α ($90° < \alpha < 180°$) with the horizontal axis (Fig. 72). Determine the vertical force which has to be applied at the point A to balance the scales (the equilibrium position coincides with the horizontal axis).

Solution. For the equilibrium state the sum of moments of all the forces acting on the scales is equal to zero, i.e. $\vec{M}_F + \vec{M}_Q = \vec{0}$.

The *moment* $\vec{M}_F$ *of a force* $\vec{F}$ *about a point* O is defined as a vector satisfying the following conditions:

(1) $|\vec{M}_F| = |\overrightarrow{OA}| \cdot |\vec{F}| \cdot \sin(\widehat{\overrightarrow{OA}, \vec{F}})$, where A is the point of application of the force $\vec{F}$;

(2) the vector $\vec{M}_F$ is perpendicular to the vectors $\vec{F}$ and $\overrightarrow{OA}$ and is directed so that if we view along it, then the smallest turn from $\overrightarrow{OA}$ to $\vec{F}$ is seen in the positive direction.

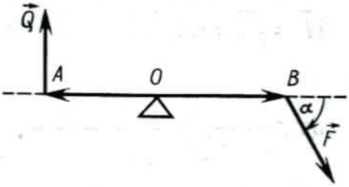


Fig. 72

Thus, in the given problem $\vec{M}_F$ is directed perpendicular to the plane of the drawing, i.e. "to us", and $\vec{M}_Q$ "from us".

Thus, $\vec{M}_F \uparrow\downarrow \vec{M}_Q$ and for the equation $\vec{M}_F + \vec{M}_Q = 0$ to be fulfilled, it is sufficient that $|\vec{M}_F| = |\vec{M}_Q|$, i.e. $a \cdot |\vec{F}| \cdot \sin\alpha = b \cdot |\vec{Q}| \cdot \sin 90°$, whence $|\vec{Q}| = \frac{a}{b} \cdot \sin\alpha$.

The moment of force considered in this problem is a vector obtained as a result of some operation on vectors.

Definition. The *vector product* of **a** vector $\vec{a}$ by a vector $\vec{b}$ is a third vector $\vec{c}$ ($\vec{c} = \vec{a} \times \vec{b}$, or $\vec{c} = [\vec{a}\vec{b}]$), obtained according to the following rule:

(1) $|\vec{c}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\widehat{\vec{a}, \vec{b}})$, i.e. the length of the vector product is equal to the product of the lengths of the vectors $\vec{a}$ and $\vec{b}$ by the sine of the angle between them;

(2) $\vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b}$, i.e. the vector product is perpendicular to either of the vectors $\vec{a}$ and $\vec{b}$;

(3) the vector $\vec{c} = \vec{a} \times \vec{b}$ is directed so that the three vectors have the same orientation as the given axes Ox, Oy, Oz.

If one of the vectors $\vec{a}$ or $\vec{b}$ is equal to zero, we suppose that $\vec{a} \times \vec{b} = \vec{0}$.

The vector product is widely used in physics. The vector product $\vec{a} \times \vec{b}$ may be interpreted as the moment of a force $\vec{b}$ about a point O, or as the linear velocity of a point rotating at an angular velocity $\vec{a}$ and lying at the terminal point of the vector $\vec{b}$, and so on.

Let us find out the geometric sense of the vector product. Since the area of the parallelogram is equal to the product of the lengths of two of its adjacent sides and the sine of the angle between them, the length of the vector product is equal to the area of the parallelogram constructed on the co-factor vectors reduced to a common origin.

Theorem. *The vector product of two non-zero vectors is equal to zero if and only if they are collinear.*

The proof is obvious.

The vector product possesses the following three arithmetic properties.

1°. $[\vec{a}\vec{b}] = -[\vec{b}\vec{a}]$, i.e. *if the factors are interchanged, the vector product reverses its sign.*

Proof. Since $(\widehat{\vec{a}, \vec{b}}) = (\widehat{\vec{b}, \vec{a}})$, then $|\,[\vec{a}\vec{b}]\,| = |\,\vec{b}\vec{a}\,|$. On the other hand, $[\vec{a}\vec{b}] \uparrow\downarrow [\vec{b}\vec{a}]$. Therefore, $[\vec{a}\vec{b}] = -[\vec{b}\vec{a}]$.

2°. $[(\alpha\vec{a})\,\vec{b}] = \alpha\,[\vec{a}\vec{b}]$ (*associative property with respect to numerical factors*).

Proof. For $\alpha = 0$ the equality is obvious. If $\alpha \neq 0$, then

$$|[(\alpha\vec{a})\,\vec{b}]| = |\alpha\vec{a}| \cdot |\vec{b}| \cdot \sin((\widehat{\alpha\vec{a}), \vec{b}}) = |\alpha| \cdot |\vec{a}| \cdot |\vec{b}| \cdot \sin((\widehat{\alpha\vec{a}), \vec{b}}).$$

For $\alpha > 0$ we have $((\widehat{\alpha\vec{a}), \vec{b}}) = (\widehat{\vec{a}, \vec{b}})$ and $|\,\alpha\,| = \alpha$; then

$$|[(\alpha\vec{a})\,\vec{b}]| = \alpha \cdot |\vec{a}| \cdot |\vec{b}| \cdot \sin(\widehat{\vec{a}, \vec{b}}) = |\alpha\,[\vec{a}\vec{b}]|.$$

For $\alpha < 0$ we have $((\widehat{\alpha\vec{a}), \vec{b}}) = \pi - (\widehat{\vec{a}, \vec{b}})$ and $|\,\alpha\,| = -\alpha$; then

$$|[(\alpha\vec{a})\,\vec{b}]| = |\alpha\vec{a}| \cdot |\vec{b}| \cdot \sin[\pi - (\vec{a}, \vec{b})] = -\alpha \cdot |\vec{a}| \cdot |\vec{b}| \cdot \sin|\widehat{\vec{a}, \vec{b}}| = |\alpha\,[\vec{a}\vec{b}]|.$$

Thus, the vectors $[(\alpha\vec{a})\,\vec{b}]$ and $\alpha\,[\vec{a}\vec{b}]$ have equal lengths.

Let us show that $[(\alpha\vec{a})\,\vec{b}] \uparrow\uparrow \alpha[\vec{a}\vec{b}]$.

If $\alpha > 0$, then $\vec{a} \uparrow\uparrow \alpha\vec{a}$ and therefore $[(\alpha\vec{a})\,\vec{b}] \uparrow\uparrow [\vec{a}\vec{b}]$.

If $\alpha < 0$, then $\alpha\vec{a} \downarrow\uparrow \vec{a}$, $[(\alpha\vec{a})\,\vec{b}] \uparrow\downarrow [\vec{a}\vec{b}]$; hence, $\alpha\,[\vec{a}\vec{b}] \uparrow\uparrow$ $\uparrow\uparrow [(\alpha\vec{a})\,\vec{b}]$.

3° $[(\vec{a}+\vec{b})\,\vec{c}] = [\vec{a}\vec{c}] + [\vec{b}\vec{c}]$ *(distributive property with respect to the sum of the vectors).*

We leave the proof of this property to the reader.

Corollary. *To carry out vector multiplication of two linear combinations it is sufficient to multiply each vector of the first combination by each vector of the second combination and to add the results thus obtained. In doing so it is necessary to retain the order of the vector factors in each separate product:*

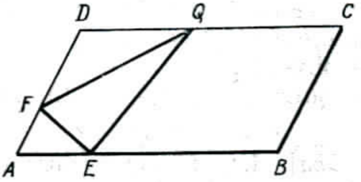


Fig. 73

$$(\alpha\vec{a}+\beta\vec{b})\times(\gamma\vec{c}+\delta\vec{d}) = \alpha\gamma\vec{a}\times\vec{c}+\beta\gamma\vec{b}\times\vec{c}+\alpha\delta\vec{a}\times\vec{d}+\beta\delta\vec{b}\times\vec{d}.$$

Problem. The area of the parallelogram $ABCD$ is equal to S. Points E, F, and Q lie on the sides $[AB]$, $[AD]$, and $[DC]$, respectively and divide them in the following ratios: $|AE| : |EB| = 1:3$, $|AF| : |FD| = 3:5$, $|DQ| : |QC| = 3:7$. Find the area of the triangle EFQ (Fig. 73).

Solution. Denoting $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$, we have

$$\overrightarrow{AE} = \frac{1}{4}\vec{a},\ \overrightarrow{AF} = \frac{3}{8}\vec{b},\ \overrightarrow{FD} = \frac{5}{8}\vec{b},\ \overrightarrow{DQ} = \frac{3}{10}\vec{a},$$

$$\overrightarrow{EF} = \overrightarrow{AE} - \overrightarrow{AF} = \frac{1}{4}\vec{a} - \frac{3}{8}\vec{b},$$

$$\overrightarrow{FQ} = \overrightarrow{FD} + \overrightarrow{DQ} = \frac{5}{8}\vec{b} + \frac{3}{10}\vec{a}.$$

The area of the triangle EFQ is equal to half the area of the parallelogram constructed on the vectors $\overrightarrow{FE}$ and $\overrightarrow{FQ}$ as sides; hence, it is equal to half the length of the vector product of these vectors:

$$S_{EFQ}=\frac{1}{2}\,|\overrightarrow{FE}\times\overrightarrow{FQ}|.$$

We then find

$$\overrightarrow{FE}\times\overrightarrow{FQ}=\left(\frac{1}{4}\vec{a}-\frac{3}{8}\vec{b}\right)\times\left(\frac{5}{8}\vec{b}+\frac{3}{10}\vec{a}\right)=$$
$$=\frac{5}{32}\vec{a}\times\vec{b}-\frac{3}{10}\vec{a}\times\vec{a}-\frac{15}{64}\vec{b}\times\vec{b}-\frac{9}{80}\vec{b}\times\vec{a}=$$
$$=\frac{5}{32}\vec{a}\times\vec{b}+\frac{9}{80}\vec{a}\times\vec{b}=\frac{43}{160}\vec{a}\times\vec{b}.$$

Since $|\vec{a}\times\vec{b}|=S$, we finally have

$$S_{EFQ}=\frac{43}{320}S.$$

PROBLEMS TO CHAPTER 4

1. Given vectors $\vec{a}$ and $\vec{b}$ construct $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $\vec{b}-\vec{a}$, $-\vec{a}-\vec{b}$.

2. Given: $|\vec{a}|=13$, $|\vec{b}|=19$, $|\vec{a}+\vec{b}|=24$. Find $|\vec{a}-\vec{b}|$.

3. Given: $|\vec{a}|=11$, $|\vec{b}|=23$, $|\vec{a}-\vec{b}|=30$. Find $|\vec{a}+\vec{b}|$.

4. The vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicular, $|\vec{a}|=5$, $|\vec{b}|=12$. Find $|\vec{a}+\vec{b}|$, $|\vec{a}-\vec{b}|$.

5. What condition must be satisfied by the vectors $\vec{a}$ and $\vec{b}$ if the vector $\vec{a}+\vec{b}$ bisects the angle between them?

6. Three forces $\vec{P}$, $\vec{Q}$, and $\vec{R}$ applied to one point have mutually perpendicular directions. Find the resultant force and the angles formed by it with the three components if $|\vec{P}|=|\vec{Q}|=|\vec{R}|=a$.

7. In a triangle ABC the following equation is fulfilled: $|\overrightarrow{CA} + \overrightarrow{CB}| = |\overrightarrow{CA} - \overrightarrow{CB}|$. Prove that the triangle ABC is right-angled.

8. Given a parallelogram $ABCD$. Prove that for an arbitrary point O in space $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$.

9. In a parallelepiped $ABCDA_1B_1C_1D_1$ the following vectors are given: $\overrightarrow{AD} = \vec{d}$, $\overrightarrow{AC} = \vec{c}$, $\overrightarrow{AB} = \vec{f}$. Express the edges of the parallelepiped in terms of these vectors.

10. Given in a triangle ABC are the medians $\overrightarrow{AE} = \vec{m}$ and $\overrightarrow{BD} = n$. Find the sides of the triangle.

11. Points M and N are the midpoints of the line segments $[AB]$ and $[CD]$. Prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{BD}$.

12. Points M and N are the points of intersection of the medians of the faces ADB and BDC in a tetrahedron $ABCD$. Prove that the vectors $\overrightarrow{MN}$ and $\overrightarrow{AC}$ are collinear and find $|\overrightarrow{MN}| : |\overrightarrow{AC}|$.

13. Vectors $\vec{a}$ and $\vec{b}$ form an angle equal to 60°. Knowing that $|\vec{a}| = 3$ and $|\vec{b}| = 4$, compute $\vec{a}\cdot\vec{b}$, $(3\vec{a} + \vec{b})^2$, $(3\vec{a} - \vec{b})\ (\vec{a} - 3\vec{b})$.

14. Prove the validity of the equation $(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = 2\ (|\vec{a}|^2 + |\vec{b}|^2)$ and find out its geometric meaning.

CHAPTER 5

PERPENDICULARITY IN SPACE. DIHEDRAL AND POLYHEDRAL ANGLES

Sec. 31. A Perpendicular to a Plane

Definition. A straight line p is said to be *perpendicular to a plane* α if it is perpendicular to any straight line contained in the plane α, i.e. $p \perp q$ for $\forall q \subset \alpha$.

Theorem. *If a straight line p is perpendicular to two non-parallel lines contained in a plane α, then it is perpendicular to this plane.*

Given: $p \perp m \subset \alpha$, $p \perp n \subset \alpha$, $m \cap n = 0$.

Prove: $p \perp \alpha$.

Proof. Let us take an arbitrary straight line $q \subset \alpha$ (Fig. 74). We then lay off vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, $\overrightarrow{BC}$, and $\overrightarrow{AP}$ on the straight lines m, n, q, and p, respectively. The vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{BC}$ are coplanar, $\overrightarrow{OM}$ and $\overrightarrow{ON}$ being non-collinear. Thus, $\overrightarrow{BC} = \alpha\overrightarrow{OM} + \beta\overrightarrow{ON}$. By hypothesis, $\overrightarrow{AP}\cdot\overrightarrow{OM} = 0$ and $\overrightarrow{AP}\cdot\overrightarrow{ON} = 0$; hence,

$$\overrightarrow{AP}\cdot\overrightarrow{BC} = \overrightarrow{AP}\,(\alpha\overrightarrow{OM} + \beta\overrightarrow{ON}) = \alpha\overrightarrow{AP}\cdot\overrightarrow{OM} + \beta\overrightarrow{AP}\cdot\overrightarrow{ON} = 0,$$

i.e. $\overrightarrow{AP} \perp \overrightarrow{BC}$, or $p \perp q$, which was required to be proved.

Thus, perpendicularity of a straight line p to two non-intersecting lines contained in a plane α is a necessary and sufficient condition for the line p to be perpendicular to the plane α.

This condition is called the *test for perpendicularity of a straight line to a plane.*

Example. The line segment CD of length l lies outside the plane containing the triangle ABC and is perpendicular to its legs $[AC]$ and $[BC]$. Find the distance from the point D

to the midpoint E of the hypotenuse if $|AC| = b$ and $|BC| = a$.

Solution. Join D to E, and E to C (Fig. 75). Since $(CD) \perp (AC)$ and $(CD) \perp (BC)$, then $(CD) \perp (CE)$, i.e. DCE is a

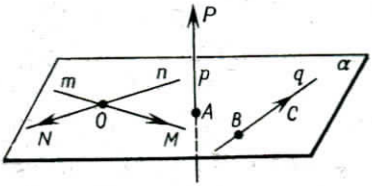


Fig. 74

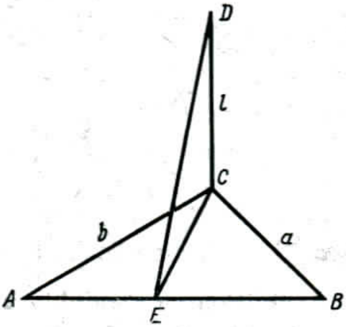


Fig. 75

right-angled triangle. In ΔCDE the leg $|CD| = l$, $|CE| = |AE| = |BE| = \frac{1}{2}|AB| = \frac{1}{2}\sqrt{a^2 + b^2}$.

Hence, $|DE| = \frac{1}{2}\sqrt{4l^2 + a^2 + b^2}$.

Sec. 32. An Inclined Line and Its Projection on a Plane. The Distance from a Point to a Plane

Definition. A straight line which cuts a plane but is not perpendicular to it is called an *inclined line* with respect to this plane (the term "inclined line" often implies a segment of this line).

A line segment joining the foot of an inclined line to the foot of the perpendicular dropped from the end-point of the segment of this line onto the same plane is obviously the orthogonal projection of the segment of an inclined line, or simply the orthogonal projection of an inclined line on this plane.

Theorem. *If from a point A not belonging to a plane α a perpendicular and inclined lines are drawn to this plane, then*

(1) *the perpendicular is shorter than any inclined line;*

(2) *congruent inclined lines have congruent projections and to congruent projections there correspond congruent inclined lines;*

(3) *of two inclined lines having non-congruent projections the one is longer whose projection is longer.*

We leave the proof of this theorem to the reader.

Suppose that F_1 and F_2 are two figures having no points in common, and $\rho(X, Y)$ are the distances between the points X belonging to F_1 and the points Y belonging to F_2. If among all the distances $\rho(X, Y)$ there exists the shortest one, then it is called the *distance between the figures* F_1 and F_2. If the figures intersect, i.e. have points in common, then the distance between them is equal to zero.

If F_1 is a plane α, and F_2 is a point A which does not belong to α, then from Theorem 1 it follows that $\rho(X, A)$ attains the least value when X is the foot of the perpendicular dropped from the point A onto the plane α. Thus, the distance from the point A lying outside α to this plane is equal to the length of the segment of the perpendicular dropped from the point A onto the plane α.

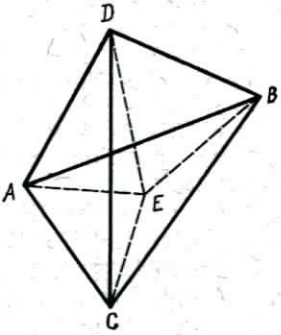


Fig. 76

Example. A point D is situated outside the plane containing the triangle ABC equidistantly from its vertices. Find the distances from the projection of the point D on this plane to the vertices of the triangle ABC, if $|AB| = |BC| = a$, and $|AC| = b$.

Solution. We join the point E which is the projection of the point D on the plane of the triangle ABC to its vertices A, B, and C (Fig. 76). The line segments $[AE]$, $[BE]$, and $[CE]$ are the projections of equal inclined lines $[AD]$, $[BD]$, and $[CD]$, and according to the above theorem, they are equal, i.e. the point E is the centre of a circle circumscribed about the triangle ABC. Since the sides of the triangle are known, the radius of the circumcircle $R = \frac{abc}{4S} = \frac{a^2b}{4S}$, where

$$S = \frac{1}{2}bh \text{ and } h = \sqrt{a^2 - \frac{b^2}{4}} = \frac{1}{2}\sqrt{4a^2 - b^2}.$$

Hence,

$$|AE| = |BE| = |CE| = R = \\ = \frac{a^2 b}{4b \cdot \frac{1}{4} \sqrt{4a^2 - b^2}} = \frac{a^2}{\sqrt{4a^2 - b^2}}.$$

Note. If a point is located outside the plane of a polygon equidistantly from its vertices, then it is orthogonally projected into the centre of the circumscribed circle.

Sec. 33. The Theorem on Three Perpendiculars

Theorem. *A straight line drawn perpendicular to the projection of an inclined line on a plane is perpendicular to the line itself.*

Given: a line $[AB]$ inclined to a plane α, its orthogonal projection $[CB]$, a line p perpendicular to (BC).

Prove: $p \perp (AB)$.

Proof. Let us draw a plane β through the inclined line $[AB]$ and its projection $[CB]$ (Fig. 77). The line p is perpendicular to the two intersecting lines (AC) and (CB) contained in the plane β. Consequently, p is perpendicular to β, in particular, $p \perp (AB)$.

Theorem (converse). *A straight line drawn perpendicular to an inclined line on a plane is also perpendicular to the orthogonal projection of this line.*

Given: an inclined line $[AB]$ to a plane, its orthogonal projection $[CB]$, a line p perpendicular to (AB).

Prove: $p \perp (CB)$.

Proof. The line p is perpendicular to two intersecting lines (AC) and (AB) lying in the plane β (Fig. 77). Hence, $p \perp \beta$ and in particular, $p \perp (CB)$.

Example 1. A point M is situated outside the plane containing the rhombus $ABCD$ equidistantly from its sides. Find the distances from the orthogonal projection of the point M on the plane of the rhombus to its sides if its altitude is equal to h.

Solution. Let O be the orthogonal projection of M on the plane $(ABCD)$ and (OE) be perpendicular to (AB) (Fig. 78). Then $[OE]$ is the projection of $[ME]$, and since $[OE]$ is perpendicular to $[AB]$, $[ME]$ is perpendicular to $[AB]$ (by the theorem on three perpendiculars). This reasoning yields the method of constructing the perpendicular drawn from the point M lying outside the plane α to a straight line contained in this plane: first drop from the projection of the point M a perpendicular onto the given line, and then join the point of their intersection to the point M. Using this

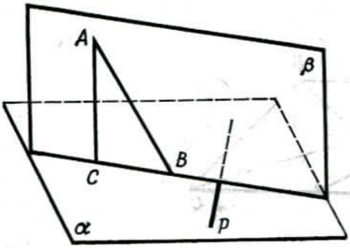


Fig. 77

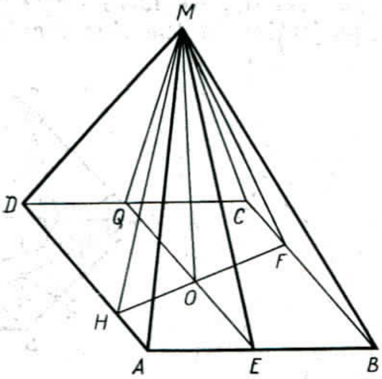


Fig. 78

method we construct $[ME]$ perpendicular to $[AB]$, $[MQ]$ perpendicular to $[DC]$, $[MH]$ perpendicular to $[AD]$, and $[MF]$ perpendicular to $[BC]$.

From the equality of the lengths of the inclined lines there follows the equality of the lengths of their orthogonal projections, i.e. $|OE| = |OF| = |OQ| = [OH]$, where $[OE]$ is perpendicular to $[AB]$, $[OF]$ is perpendicular to $[BC]$, $[OQ]$ to $[DC]$, and $[OH]$ to $[AD]$. Consequently, the point O is equidistant from the sides of the rhombus, i.e. it is the centre of the circle inscribed in it. Therefore it coincides with the point of intersection of its diagonals. But $|OE| + |OQ| = h$, hence, $|OE| = |OQ| = |OF| = |OH| = h/2$.

Note. If a point lies outside the plane of a polygon equidistantly from its sides, then its orthogonal projection coincides with the centre of the circle inscribed in this polygon.

Example 2. From the vertex A of the rectangle $ABCD$ a straight line l is drawn lying outside the rectangle, which forms equal (acute) angles with the sides $[AB]$ and $[AD]$ (Fig. 79). Into what parts is the diagonal $[BD]$ divided by the orthogonal projection of this line on the plane $(ABCD)$ if $|AB| = a$, and $|BC| = b$.

Solution. Let $F \in l$, $[FF_1] \perp (ABCD) = \alpha$, $[F_1Q] \perp (AB)$, $[F_1H] \perp (AD)$. Then $[AF_1] = \mathrm{pr}_\alpha [AF]$, $[QF_1] = \mathrm{pr}_\alpha [QF]$, $[HF_1] = \mathrm{pr}_\alpha [HF]$. By the theorem on three perpendiculars $[FQ] \perp (AB)$, $[FH] \perp (AD)$. Since $|FH| = |FQ|$ $(\triangle AFQ \cong \triangle AFH)$, we have $|F_1Q| = |F_1H|$, i.e. the orthogonal projection of the line l coincides with the bisector of the angle BAD. By the property of the bisector of an interior angle of a triangle $|DM| : |MB| = |AD| : |AB|$, where $|AD| = b$, $|AB| = a$ and $|DM| + |MB| = |DB| = \sqrt{a^2 + b^2}$. Hence,

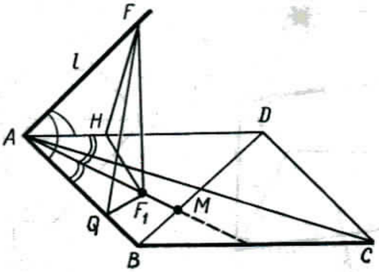


Fig. 79

$$\frac{|DM|}{|MB|} = \frac{b}{a}, \quad \frac{|DM| + |MB|}{|MB|} = \frac{a+b}{a},$$

$$\frac{\sqrt{a^2+b^2}}{MB} = \frac{a+b}{a},$$

whence

$$|MB| = \frac{a\sqrt{a^2+b^2}}{a+b} \text{ and } |DM| = \frac{b\sqrt{a^2+b^2}}{a+b}.$$

Note. The solution of this problem proves that *if an inclined line forms equal angles with two straight lines on a plane passing through the trace of the inclined line on this plane, then its orthogonal projection is the bisector of the angle formed by these straight lines.*

Sec. 34. The Angle Between an Inclined Line and a Plane

Definition. *The angle between an inclined line p and a plane* α is defined as the magnitude of the acute angle between this line and its orthogonal projection on this plane (Fig. 80).

This angle possesses the following property: of all the angles formed by the inclined line with any straight line contained in the plane it is the smallest one. Let us prove it.

Theorem. *The angle between an inclined line and its orthogonal projection on a plane is smaller than any angle formed by this inclined line with any straight line drawn in this plane through the trace of the inclined line* (Fig. 81).

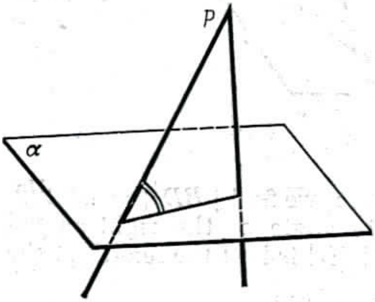


Fig. 80

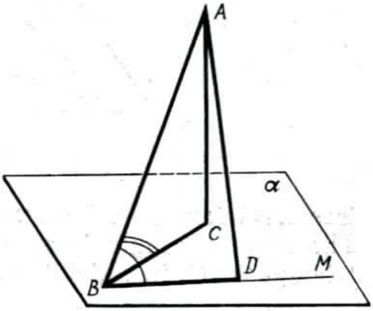


Fig. 81

Given: a line $[AB]$ inclined to a plane α, $[AC]$ is perpendicular to α, $[BC] = \mathrm{pr}_\alpha\,[AB]$, $[BM] \subset \alpha$.

Prove: $\widehat{ABC} < \widehat{ABM}$.

Proof. We lay off on (BM) a line segment $[BD] = [BC]$, join the points A and D, and consider the triangles ABC and ABD thus obtained. In them $[AB]$ is a common side, and $[BC] \cong [BD]$, i.e. two sides of one triangle are congruent to two sides of the other triangle, while the third sides are non-congruent, and $|AD| > |AC|$. From this it follows that $\widehat{ABM} > \widehat{ABC}$.

Example. The leg $[AC]$ of an isosceles right-angled triangle ABC lies in a plane α, and the leg $[BC]$ is inclined to it at an angle of 45°. Find the angle formed by the hypotenuse $[AB]$ with the plane α (Fig. 82).

Solution. Let $[BD]$ be perpendicular to α, and $D = [BD] \cap \alpha$. Then $[CD] = \mathrm{pr}_\alpha [BC]$, $[AD] = \mathrm{pr}_\alpha [AB]$, $\widehat{BCD} = 45°$, $\widehat{BAD}$ being the required angle. If $|AC| = |BC| = a$, then $|AB| = a\sqrt{2}$. From the right-angled

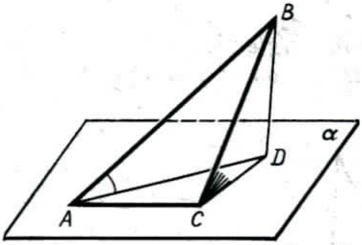


Fig. 82

isosceles triangle BCD ($|BC| = a$) we find $|BD| = a\sqrt{2}/2$. The required angle is an acute angle of the right-angled triangle ABD, in which the leg $[BD]$ is half the length of the hypotenuse $[AB]$. Hence, $\widehat{BAD} = 30°$.

Sec. 35. Dependence Between Parallelism and Perpendicularity of Straight Lines and Planes

Theorem 1. *A plane perpendicular to one of two parallel lines is perpendicular to the other.*

Given: $p \parallel q$, $p \perp \alpha$.

Prove: $q \perp \alpha$.

Proof. In the plane α we draw two intersecting lines m and n (Fig. 83). Since $p \parallel q$, we have $\widehat{(p, n)} = \widehat{(q, n)}$ and $\widehat{(p, m)} = \widehat{(q, m)}$ (see Sec. 18). By hypothesis, $\widehat{(p, n)} = \widehat{(p, m)} = 90°$, therefore $\widehat{(q, n)} = \widehat{(q, m)} = 90°$, and, consequently, q is perpendicular to α (by Theorem of Sec. 31).

Theorem 2 (converse). *Straight lines perpendicular to one plane are parallel to each other.*

Given: $p \perp \alpha$, $q \perp \alpha$.
Prove: $p \parallel q$.

Proof. Suppose $p \nparallel q$. Through an arbitrary point $A \in q$ we draw a straight line r parallel to p (Fig. 84). According to Theorem 1 r is perpendicular to α. Thus, through the point A there pass two different lines q and r perpendicular to the plane α, which is false.

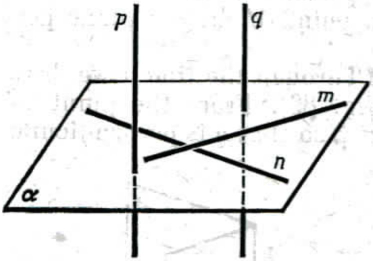


Fig. 83

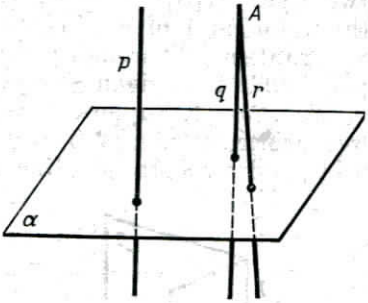


Fig. 84

Theorem 3. *A straight line perpendicular to one of two parallel planes is also perpendicular to the other plane.*

Given: $\alpha \parallel \beta$, $p \perp \alpha$.

Prove: $p \perp \beta$. *Proof*. Through the line p we draw two intersecting planes γ and δ (Fig. 85). Let $(AB) = \alpha \cap \gamma$, $(BC) = \alpha \cap \delta$, $(A_1B_1) = \beta \cap \gamma$, $(B_1C_1) = (\beta, \delta)$. Then $(AB) \parallel (A_1B_1)$, $(BC) \parallel (B_1C_1)$ (by Theorem 2 of Sec. 17). Since $p \perp (AB)$ and $p \perp (BC)$, then $p \perp (A_1B_1)$ and $p \perp (B_1C_1)$, and hence, $p \perp \beta$.

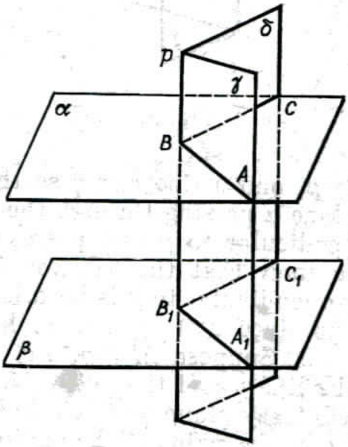


Fig. 85

Theorem 4 (converse). *Two planes perpendicular to one and the same line are parallel to each other.*

Given: $\alpha \perp p$, $\beta \perp p$.

Prove: $\alpha \parallel \beta$.

Proof. We carry out the same construction as in Theorem 3 (see Fig. 85). Since p is perpendicular to (A_1B_1) and (AB), and (AB) and (A_1B_1) are contained in the plane γ, then (AB) is parallel to (A_1B_1). We prove in a similar way that (BC) is parallel to (B_1C_1). Thus, two intersecting lines (AB) and (BC) contained in the plane α are respectively parallel to two lines (A_1B_1) and (B_1C_1) lying in the plane β. Hence (by Theorem 1 of Sec. 17) α is parallel to β.

Problem 1. Through a given point M draw a plane perpendicular to a given line p.

Solution. 1. Suppose $M \in p$. Through the line p we draw two arbitrary planes $\beta \nparallel \gamma$ (Fig. 86). From the point M we then draw a straight line $q \subset \beta$ so that q is perpendicular to p, and a line $r \subset \gamma$ so that r is perpendicular to p. The plane α passing through the intersecting lines q and r is perpendicular to p and passes through $M = q \cap r$. It is easy to show that the solution is unique. Indeed, any plane δ perpendicular to p is parallel to the plane α (by Theorem 4), and since $\alpha \cap \delta \ni M$, we have $\delta = \alpha$.

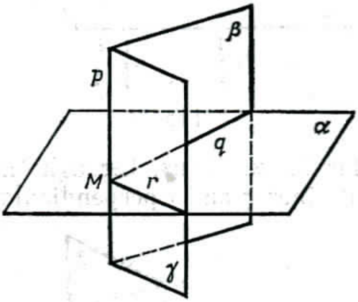


Fig. 86

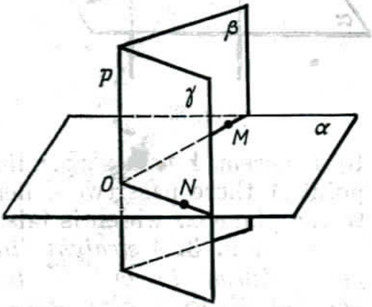


Fig. 87

2. Suppose $M \notin p$. Let β denote the plane containing the line p and the point M. Through the line p we draw an arbitrary plane γ which does not coincide with the plane β. From the point M belonging to the plane β we then draw (MO) perpendicular to p and from the point O lying in the plane γ a line (ON) perpendicular to p (Fig. 87). The plane passing through two intersecting lines (MO) and (ON) is perpendicular to p. Its uniqueness is proved as in the above case.

Problem 2. Through a given point M draw a straight line perpendicular to a given plane α.

Solution. In the given plane α we take an arbitrary line q and draw a plane β perpendicular to q through the point M (Problem 1). In this plane we then draw through the point M a straight line $p \perp \alpha \cap \beta$ (Fig. 88). Since $p \perp q \subset \alpha$ and $p \perp \alpha \cap \beta \subset \alpha$, the line p is perpendicular to α, and $p \ni M$.

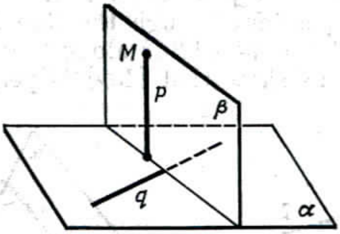


Fig. 88

The solution is unique: indeed, any other line r perpendicular to α is parallel to p (by Theorem 2), and since, moreover, $M \in r \cap p$, then $r = p$.

Sec. 36. The Distance Between Skew Lines

Theorem. *The distance between skew lines p and q is equal to the length of the perpendicular dropped from any point on one line onto the plane passing through the other line parallel to the first line.*

Given: skew lines p and q, $p \subset \alpha \parallel q$, $A \in q$, $[AB] \perp \alpha$, $B \in \alpha$.

Prove: $\rho(p, q) = |AB|$.

Proof. We take arbitrary points $C \in q$ and $D \in p$, and draw from the point C a line segment $[CF]$ perpendicular to α (Fig. 89). Since $[CD]$ is an inclined line, and $[CF]$ is a segment of a perpendicular, we have $[CD] \geqslant [CF]$, or $[CD] \geqslant [AB]$, the equality being obtained when D coincides with F.

According to the definition of distance (see Sec. 32), this means that $\rho(p, q) = |AB|$.

Example. Through the vertex A of an isosceles triangle ABC there passes a straight line l which forms equal angles with the sides $[AB]$ and $[AC]$, and an angle of 60° with the plane containing the triangle (Fig. 90). Det ermine the distance between the lines l and $[BC]$ if $|BC| = a$.

Solution. Since the inclined line forms equal angles with the sides $[AB]$ and $[AC]$, its projection coincides with the bisector of the angle BAC, i.e. with $[AD]$ (see Example 2 of Sec. 32). In the plane β passing through the lines l and $[AD|$, we draw $[DQ]$ perpendicular to l.

The plane α passing through the line l and parallel to $[BC]$ intersects the plane (ABC) along the line (MN) parallel to (BC). Let us show that $[DQ]$ is perpendicular to α.

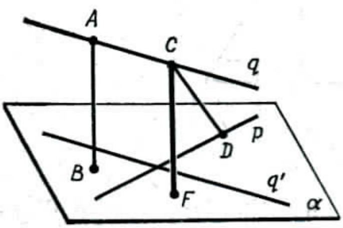


Fig. 89

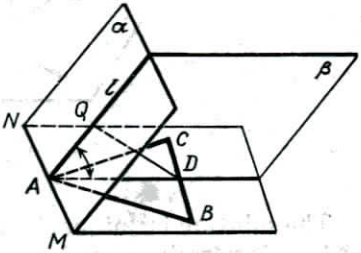


Fig. 90

By construction, $[DQ]$ is perpendicular to l, and by the theorem on three perpendiculars, $[DQ]$ is perpendicular to (BC). Since (BC) is parallel to $[MN]$, then $[DQ]$ is perpendicular to $[MN]$. Hence, $|DQ| \perp \alpha$ is the required distance between the lines l and $[BC]$. To find it let us consider the right-angled triangle ADQ, in which $[AD] = a\sqrt{3}/2$, and $\widehat{QAD} = 60°$. Consequently, $|DQ| = |AD| \sin 60° = 3a/4$.

Sec. 37. The Triple Product of Three Vectors. The Test for Coplanarity of Three Vectors

In Chapter 4, with the aid of the scalar and vector products, we obtained the tests for collinearity and perpendicularity of two vectors. Combining these two products we can get the test for coplanarity of three vectors.

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three non-zero vectors. We take the first two vectors $\vec{a}$ and $\vec{b}$, and write a vector product $\vec{a} \times \vec{b}$, which is then scalarly multiplied by the vector $\vec{c}$. As a result, we get a number which is called the *triple product* of the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, and is denoted $\vec{a}\vec{b}\vec{c}$ (we remind here of the fact that the first two vectors are multiplied as vectors and the result is then multiplied by the third vector scalarly).

Let us find out under which conditions the triple product of three non-zero vectors is equal to zero.

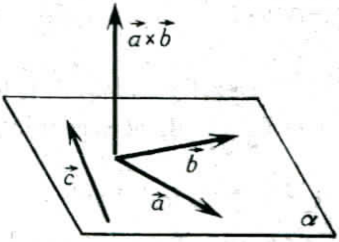


Fig. 91

Theorem (the test for coplanarity of three vectors). *Three non-zero vectors are coplanar if and only if their triple product is equal to zero.*

1. Given: $\vec{a} \neq \vec{0}$, $\vec{b} \neq 0$, $\vec{c} \neq \vec{0}$ are coplanar vectors.

Prove: $\vec{a}\vec{b}\vec{c} = 0$.

Proof. If $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} \times \vec{b} = \vec{0}$ (see Sec. 30) and then $\vec{a}\vec{b}\vec{c} = 0$. If $\vec{a} \nparallel \vec{b}$, then $\vec{a} \times \vec{b} \neq \vec{0}$ and the vector $\vec{a} \times \vec{b}$ is perpendicular to the plane α, containing the vectors $\vec{a}$ and $\vec{b}$ (Fig. 91). In particular, $[\vec{a}\times\vec{b}] \perp \vec{c}$ and, consequently, the scalar product of the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$ is equal to zero, i.e. $\vec{a}\vec{b}\vec{c} = 0$.

2. Given: $\vec{a}\vec{b}\vec{c} = 0$, $\vec{a} \neq 0$, $\vec{b} \neq 0$, $\vec{c} \neq 0$.

Prove: $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

Proof. The scalar product of the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$ is equal to zero when either $\vec{a} \times \vec{b} = \vec{0}$ $(\vec{c} \neq \vec{0})$, or $(\vec{a} \times \vec{b}) \perp \perp \vec{c}$.

If $\vec{a} \times \vec{b} = 0$, then $\vec{a} \parallel \vec{b}$ (see Sec. 30) and, consequently, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar. But if $(\vec{a} \times \vec{b}) \perp \vec{c}$, then $\vec{c} \subset \alpha$ (Fig. 91), and hence, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

Example. Prove that the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, satisfying the condition $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$, are coplanar.

Solution. Let us show that $\vec{a}\vec{b}\vec{c} = 0$. To this end we multiply scalarly both members of the given equation by the vector $\vec{c}$. We get

$$[\vec{a}\vec{b}]\cdot\vec{c} + [\vec{b}\vec{c}]\cdot\vec{c} + [\vec{c}\vec{a}]\cdot\vec{c} = \vec{0}\cdot\vec{c}, \text{ or } \vec{a}\vec{b}\vec{c} + \vec{b}\vec{c}\vec{c} + \vec{c}\vec{a}\vec{c} = 0.$$

Since $[\vec{b}\vec{c}] \perp \vec{c}$, $[\vec{c}\vec{a}] \perp \vec{c}$, then $\vec{b}\vec{c}\vec{c} = 0$, $\vec{c}\vec{a}\vec{c} = 0$ and, hence, $\vec{a}\vec{b}\vec{c} = 0$, which was required to be proved.

Sec. 38. Dihedral Angles

Definition 1. A convex figure bounded by two half-planes α and β emanating from one straight line $(MN) = \alpha \cap \beta$ is called the *dihedral angle* (Fig. 92).

The half-planes α and β are called the *faces*, and their common line (MN) the *edge* of a dihedral angle.

The set of all points of a dihedral angle which do not belong to its faces form the inside domain of this angle.

A dihedral angle is denoted in the following way: $\angle \alpha MN \beta$, or $\alpha p \beta$ (p is the edge), or, briefly, by its edge $\angle MN$, or $\angle p$.

As is obvious, a dihedral angle is the intersection of two half-spaces bounded by two non-parallel planes α and β.

Definition 2. The intersection of a dihedral angle with a plane perpendicular to its edge is called the *plane angle* of the dihedral angle: $\angle ABC = \gamma \cap (\angle MN)$ (Fig. 93).

Or in other words: A plane angle of a dihedral angle is an angle formed by the two rays which are the intersections of the faces of the dihedral angle and a plane perpendicular to the edge.

For the angle ABC to be a plane angle of the dihedral angle it is necessary and sufficient that (AB) and (BC) are perpendicular to the edge (MN). It should be remembered that the magnitude of the plane angle does not depend on the choice of the point B on the edge (MN).

Indeed, all planes perpendicular to the edge of a dihedral angle intersect its faces α and β along the rays of the same direction (Fig. 94, *b*); consequently, all the plane angles of a given dihedral angle are congruent, their magnitudes being equal.

Definition 3. A *measure of a dihedral angle* is a measure of one of its plane angles.

Theorem. *Plane angles corresponding to congruent dihedral angles are congruent, and conversely, to congruent plane angles there correspond congruent dihedral angles.*

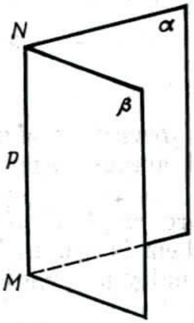


Fig. 92

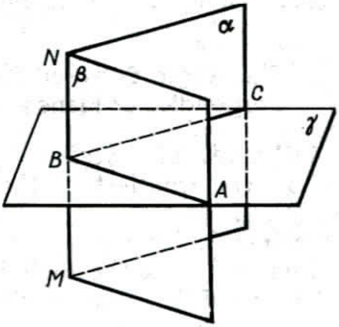
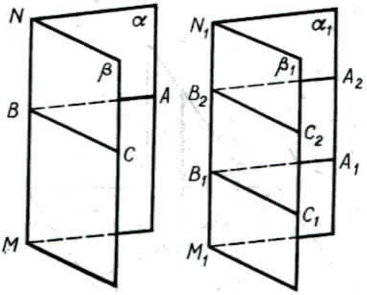


Fig. 93

Proof. 1. Suppose that the angle MN is congruent to the angle M_1N_1 (Fig. 94). Let us consider the displacement in which the angle MN coincides with the angle M_1N_1, and the angle ABC coincides with an angle $A_2B_2C_2$. Since a displacement preserves orthogonality, (A_2B_2) is perpendicular to (M_1N_1), and (C_2B_2) to (M_1N_1), i.e. the angle $A_2B_2C_2$ is the plane angle of the dihedral angle M_1N_1 and, hence, is congruent to another of its plane angles $A_1B_1C_1$. Thus, $\angle ABC \cong \angle A_2B_2C_2$, $\angle A_2B_2C_2 \cong \angle A_1B_1C_1$; therefore, $\angle ABC \cong \angle A_1B_1C_1$.



(a) (b)

Fig. 94

2. Let the angle ABC be congruent to the angle $A_1B_1C_1$. These angles are plane angles of the dihedral angles MN and M_1N_1, respectively. Since $\angle ABC \cong \angle A_1B_1C_1$, there exists a displacement which brings (AB) into coincidence with (A_1B_1), and (BC) with (B_1C_1). Moreover, the plane (ABC) coincides with the plane $(A_1B_1C_1)$, and the edge (MN) with the edge (M_1N_1) (indeed, (MN) is perpendicular to

(ABC), and (M_1N_1) to $(A_1B_1C_1)$). Therefore, the plane α passing through the edge (MN) and point A will coincide with the plane α_1, and the plane β with the plane β_1. Thus, the angle $\alpha MN\beta$ will coincide with the angle $\alpha_1M_1N_1\beta_1$.

Sec. 39. The Angle Between Planes. Perpendicular Planes

Definition 1. The *angle between two intersecting planes* is defined as the smallest of the dihedral angles formed by these planes.

Definition 2. Two planes α and β are *perpendicular* to each other ($\alpha \perp \beta$) if the angle between them is equal to 90°.

Let us consider the properties of mutually perpendicular planes.

Theorem 1. *A plane α passing through a perpendicular p to the other plane β is perpendicular to the plane β.*

Given: $p \subset \alpha$, $p \perp \beta$.

Prove: $\alpha \perp \beta$ (Fig. 95).

Proof. Let $B = p \cap \beta$. The planes α and β intersect along a straight line $(\alpha \cap \beta) \ni B$. Through the point B in the plane β we draw a line q perpendicular to $(\alpha \cap \beta)$. Since $(\alpha \cap \beta)$ is perpendicular to p, and to q, the angle $\widehat{(p, q)} = 90°$ is a plane angle of the dihedral angle $\alpha \cap \beta$. Hence, α is perpendicular to β.

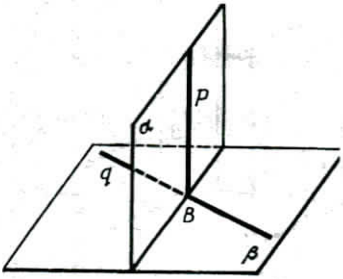


Fig. 95

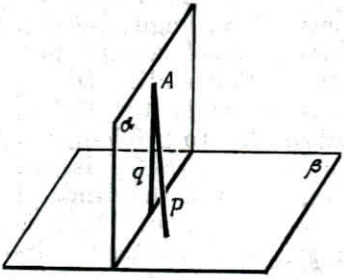


Fig. 96

Theorem 2. *If two planes α and β are mutually perpendicular, then a straight line p drawn in the plane α perpendicu-*

lar to the line of intersection of the planes α *and* β *is perpendicular to the plane* β.

Given: $\alpha \perp \beta$, $p \subset \alpha$, $p \perp (\alpha \cap \beta)$.

Prove: $p \perp \beta$.

Proof. Let $p \cap \beta = B$. Through the point B we draw $q \subset \beta$ and $q \perp (\alpha \cap \beta)$ (Fig. 95); (p, q) is a plane angle of the right dihedral angle $\alpha p \beta$. Hence, $\widehat{(p, q)} = 90°$, i.e. $p \perp q$. Since $p \perp (\alpha \cap \beta)$ and $p \perp q$, then $p \perp \beta$.

Corollary. *If* α *and* β *are two mutually perpendicular planes and from a point* A *contained in the plane* α *a perpendicular* p *is dropped onto the plane* β, *then it lies in the plane* α.

Given: $\alpha \perp \beta$, $p \perp \beta$, $A \in p \cap \alpha$.

Prove: $p \subset \alpha$

Proof. From the point $A \in \alpha$ we draw q perpendicular to $(\alpha \cap \beta)$. According to Theorem 2, q is perpendicular to β. If we suppose that $p \not\subset \alpha$ (Fig. 96), then $p \neq q$, and $p \cap q = A$. But this means that two perpendiculars (p and q) are dropped from the point A onto the plane α, which is impossible.

Example. A straight line (AB) passes through two points A and B lying in two mutually perpendicular planes α and β, respectively. The perpendiculars drawn from the points A and B to the line $\alpha \cap \beta$ have lengths equal to a and b, respectively. Determine the length of the line segment $[AB]$ and the lengths of its projections on the given planes if the distance between the feet of the perpendiculars is equal to c (Fig. 97).

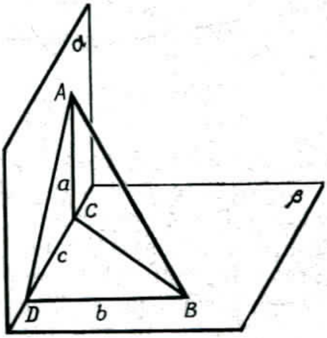


Fig. 97

Solution. Obviously, $[AD] = \mathrm{pr}_{\alpha}\,[AB]$, $[CB] = \mathrm{pr}_{\beta}[AB]$. To find $|AD|$, $|AB|$, and $|BC|$ consider $\triangle ACD$ and $\triangle ADB$. In a right-angled $\triangle ACD$ (here $(AC) \perp \alpha \cap \beta$) the legs $[AC]$ and $[CD]$ are known. Therefore, $|AD| = \sqrt{|AC|^2 + |CD|^2} = \sqrt{a^2 + c^2}$. Now in the right-angled $\triangle ADB$ we know the legs $[AD]$ and $[DB]$. Hence,

$|AB| = \sqrt{|AD|^2 + |DB|^2} = \sqrt{a^2 + b^2 + c^2}$. $|BC|$ is found from the right-angled $\triangle ACB$ $[(AC) \perp \beta]$:

$$|BC| = \sqrt{|AB|^2 - |AC|^2} = \sqrt{c^2 + b^2}.$$

Sec. 40. Orthogonal Projecting

Orthogonal projecting is a particular case of parallel projecting in which the direction of projecting is perpendicular to the plane of projection (see Sec. 19). Therefore, orthogonal projecting possesses all the properties of parallel projecting, i.e.:

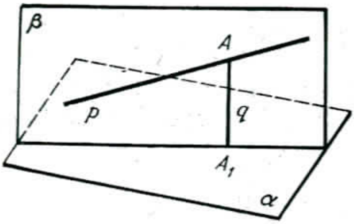


Fig. 98

(1) an orthogonal projection of a straight line is either a point, or a straight line;

(2) orthogonal projections of parallel lines not perpendicular to the plane of projection are parallel;

(3) the ratio of the lengths of the segments of a straight line p is equal to that of their orthogonal projections.

The plane β passing through the line p perpendicular to the plane of projection α is a projecting plane (see Sec. 19). This plane β passes through the line p and the perpendicular q drawn from an arbitrary point $A \in p$ to the plane α (Fig. 98). Indeed, according to Theorem 1 of Sec. 39, the plane β is perpendicular to the plane α, and from the corollary of Sec. 35 it follows that all the perpendiculars dropped from the points of the line p onto the plane α lie in the plane β. It is obvious, that the projecting plane is uniquely determined by the straight line p if p is not perpendicular to the plane of projection.

Orthogonal projecting is a mostly often used kind of parallel projecting, therefore the term "projection" should henceforth be understood as "orthogonal projection".

The method of orthogonal projecting is widely used in engineering drawing, which is based on *Monges method* consisting in orthogonally projecting a figure on two mutually perpendicular planes: the horizontal plane H, and the vertical plane V.

Example 1. Given: a point M_1 in the horizontal plane of projection, and a point M_2 in the vertical plane. Under what condition are M_1 and M_2 the projections of one and the same point in space?

Solution. If $M_1 = \mathrm{pr}_H M$ and $M_2 = \mathrm{pr}_V M$, then the plane γ passing through the points M_1, M_2, and M is perpendicular to the planes H and V and, consequently, is perpendicular to the line of their intersection, i.e. to the axis of projection (XX_1) (Fig. 99). Let $O = \gamma \cap (XX_1)$. It is obvious, that $[M_1O]$ is perpendicular to (XX_1) and $[M_2O]$ to (XX_1).

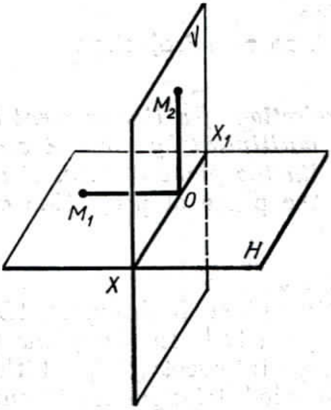


Fig. 99

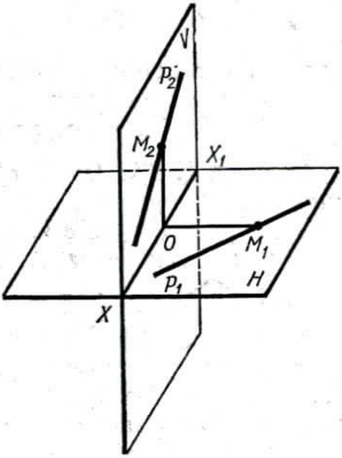


Fig. 100

Thus, if M_1 and M_2 are the projections of one and the same point M, then the perpendiculars drawn from the points M_1 and M_2 to the axis of projection (XX_1) intersect on this axis. Conversely, if M_1 belonging to H, and M_2 belonging to V satisfy the condition $(M_1O) \perp (XX_1)$ and $(M_2O) \perp (XX_1)$, then they are the projections of one and the same point in space.

Example 2. Given: the horizontal and vertical projections of a straight line p, and the vertical projection of a point M belonging to p. Find the horizontal projection of the point M.

Solution. By hypothesis, $p_1 = \mathrm{pr}_H p$, $p_2 = \mathrm{pr}_V p$, $M_2 = \mathrm{pr}_V M$, where M belongs to p. Since $M \in p$, then the required point $M_1 \in p_1$ (Fig. 100). On the other hand, M_1 and M_2, being the projections of one and the same point M, satisfy the condition $(M_1O) \perp (XX_1)$, $(M_2O) \perp (XX_1)$ (see Example 1). From this it follows that to find the point $M_1 = \mathrm{pr}_H M$ we have to draw (M_2O) from the point M_2 perpendicular to (XX_1) to intersect the axis (XX_1) at point O, and then from the point O in the plane H to draw (OM_1) perpendicular to (XX_1) to intersect $p_1 = \mathrm{pr}_H p$. Obviously, $M_1 = \mathrm{pr}_H M$.

Sec. 41. The Length of a Projection of a Line Segment. The Area of a Projection of a Plane Polygon

Theorem 1. *The length of a projection of a line segment is equal to the length of the segment multiplied by the cosine of the angle formed by the segment with the plane of projection.*

Given: a line segment $[AB]$, the plane of projection α, $\varphi = (\widehat{[AB], \alpha})$.

Prove: $|\,\mathrm{pr}_\alpha [AB]\,| = [AB] \cdot \cos \varphi$.

Proof. Let us draw a projecting plane β through (AB) (Fig. 101). Obviously, $[A_1B_1] = \mathrm{pr}_\alpha [AB]$. In the plane β we draw (AC) parallel to (A_1B_1) to intersect (BB_1) at the point C. Considering the right-angled triangle ABC and taking into account that $|A_1B_1| = |AC|$, we get

$$|A_1B_1| = |AC| = |AB| \cdot \cos \varphi.$$

Theorem 2. *The area of a projection of a plane polygon is equal to the area of the projected polygon multiplied by the cosine of the angle between the plane containing the polygon and the plane of projection.*

Proof. Let us first suppose that the projected figure is a triangle ABC, whose side, say, $[AC]$ lies in the plane of projection α (Fig. 102). Through the sides $[AB]$ and $[AC]$ we draw a plane β, which will intersect the plane of projection along a straight line (AC). From the point B we drop a perpendicular $[BD]$ onto the plane α, and from the point $D = (BD) \cap \alpha$ a perpendicular $[DF]$ onto (AC) : $(DF) \cap (AC) = F$. Joining the points B and F we get $[BF]$, which is perpendicular to (AC) (by virtue of the theorem on three perpendiculars). Thus, $[DF]$ is the altitude of the triangle ADC, and $[BF]$ is the altitude of the triangle ABC, $\angle$ BFD being a plane angle of the dihedral angle formed by the planes α and β.

Two triangles ABC and ADC have a common base $[AC]$, consequently, $S_{ADC} : S_{ABC} = |DF| : |BF|$, and since $|DF| : |BF| = \cos\varphi$, we get $S_{ADC} : S_{ABC} = \cos\varphi$, whence

$$S_{ADC} = \text{area of projection of } ABC = S_{ABC} \cdot \cos\varphi.$$

Let now the projected figure be a triangle none of whose sides lying in the plane of projection and being parallel to it (Fig. 103). In the triangle ABC through the vertex C we

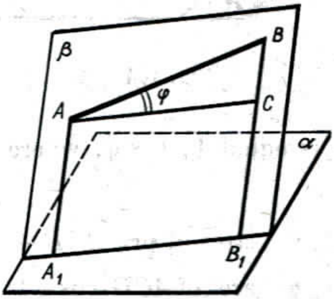


Fig. 101

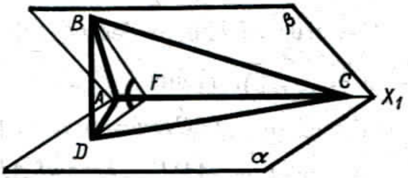


Fig. 102

draw a straight line (CD) parallel to the line $(XX_1) = (ABC) \cap \alpha$, and then through (CD) a plane parallel to α. As a result of this construction the projecting of the triangle ABC on the plane α is reduced to projecting two triangles on the plane parallel to α, with respect to which the conditions of the above considered case are satisfied. Since

$$\frac{\text{area of projection of } BDC}{\text{area of } BDC} = \cos\varphi \text{ and}$$

$$\frac{\text{area of projection of } ADC}{\text{area of } ADC} = \cos\varphi,$$

we have

$$\frac{\text{area of projection of } BDC + \text{area of projection of } ADC}{\text{area of } BDC + \text{area of } ADC} = \cos\varphi.$$

Hence,

$$\text{area of projection of } ABC = \text{area of } ABC \cdot \cos\varphi.$$

Let us finally suppose that the projected figure is any plane polygon. It can be divided by the diagonals into a number of triangles, with respect to which the theorem is proved. The ratio of the area of any of these triangles to the area of the projected triangle is equal to $\cos\varphi$, where $\varphi = \widehat{(\alpha, \beta)}$. Hence,

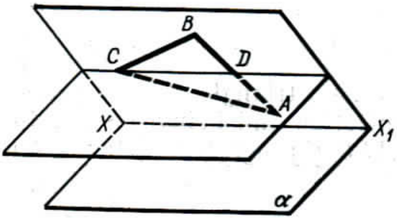


Fig. 103

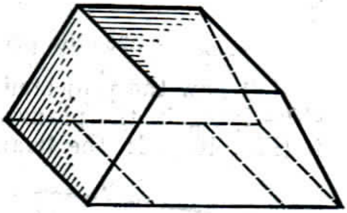


Fig. 104

$$\text{area of } \mathrm{pr}_\alpha\, ABCDE = \text{area of } \mathrm{pr}_\alpha\, ABC + \text{area of } \mathrm{pr}_\alpha\, ACD + \\ + \text{area of } \mathrm{pr}_\alpha\, ADE = \text{area of } ABC \cdot \cos\varphi + \text{area of } ACD \cdot \cos\varphi + \\ + \text{area of } ADE \cdot \cos\varphi = (\text{area of } ABC + \text{area of } ACD + \\ + \text{area of } ADE) \cdot \cos\varphi = \text{area of } ABCDE \cdot \cos\varphi.$$

Example. The area of the projection of a shed on a horizontal plane is equal to q. Find the area of the shed if the lateral rectangles are congruent and are inclined to the horizontal at an angle of 60°, and the middle portion of the shed which is a square with the side a is parallel to the horizontal plane (Fig. 104).

Solution. The projection of the shed consists of the projections of two lateral portions and the projection of the middle. The area of the projection of the middle is equal to a^2. Consequently, the area of the projections of two congruent lateral parts is equal to $q - a^2$. Whence, by Theorem 2, the area of the lateral parts is equal to $\frac{q-a^2}{\cos 60^\circ} = 2(q - a^2)$, and the total area of the shed to $2(q - a^2) + a^2 = 2q - a^2$.

Sec. 42. The Area of a Projection of an Arbitrary Plane Figure

Theorem. *The area of a projection of an arbitrary plane figure is equal to the area of the projected figure multiplied by the cosine of the angle between the plane containing the figure and the plane of projection.*

Given: a plane figure F, the plane of projection α, $\varphi = (\widehat{\text{area of } F, \alpha})$.

Prove: area of $\mathrm{pr}_\alpha F$ = area of $F \cdot \cos\varphi$.

Proof. Let us lay a squared measuring grid on the figure F, the side of the squares being equal to $1/10^n$ (Fig. 105). Suppose F contains r such squares. We denote their join by A_n. It is clear, that the area of $A_n = r/10^{2n}$ units². Let B_n denote the join of q squares containing F (in Fig. 105 this figure consists of A_n and the black squares). Obviously, $B_n = q/10^{2n}$ units². When projected, the projection of F is contained inside the projection of B_n and contains the projection of A_n: $\mathrm{pr}_\alpha A_n \subset \mathrm{pr}_\alpha F \subset \mathrm{pr}_\alpha B_n$. Therefore,

$$\text{area of } \mathrm{pr}_\alpha A_n \leqslant \text{area of } \mathrm{pr}_\alpha F \leqslant \text{area of } \mathrm{pr}_\alpha B_n,$$

where the area of $\mathrm{pr}_\alpha A_n$ = area of $A_n \cos\varphi$ and the area of $\mathrm{pr}_\alpha B_n$ = area of $B_n \cos\varphi$. These areas give the area of $\mathrm{pr}_\alpha F$ a lower and an upper bounds, and as n increases unboundedly, these bounds approach each other with no limit; i.e.

$$\lim_{n\to\infty} \text{area of } \mathrm{pr}_\alpha A_n = \lim_{n\to\infty} \text{area of } \mathrm{pr}_\alpha B_n.$$

Since

$$\lim_{n\to\infty} \text{area of } \mathrm{pr}_\alpha A_n = \lim_{n\to\infty} \text{area of } A_n \cdot \cos\varphi = \text{area of } F \cdot \cos\varphi,$$

we get

$$\text{area of } \mathrm{pr}_\alpha F = \text{area of } F \cdot \cos\varphi,$$

which was required to be proved.

Example. A circle of radius R is projected on a plane inclined to the plane containing the circle at an acute angle φ. Find the area of the projection of the circle.

Solution. The projection of a circle on a plane not parallel to it is always an ellipse (Fig. 106). In this case the diameter of the circle $[AB]$, which is parallel to the plane of projection, is projected into the major axis of the ellipse $[A_1B_1]$, and the diameter $[CD]$ which is perpendicular to

[AB], into the minor axis of the ellipse: $|A_1B_1| = 2a = 2R$, $|C_1D_1| = 2b = |CD| \cos\varphi = 2R\cdot\cos\varphi$.

According to the above theorem, the area of the projection of the circle is equal to its area πR^2 multiplied by the cosine of the angle of projection, i.e.

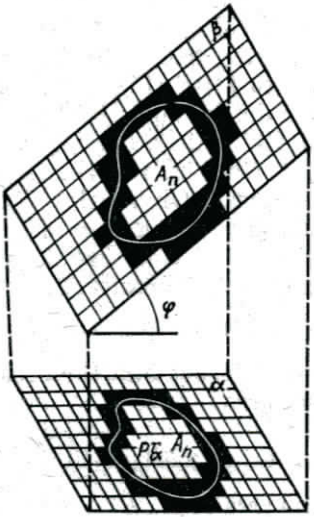


Fig. 105

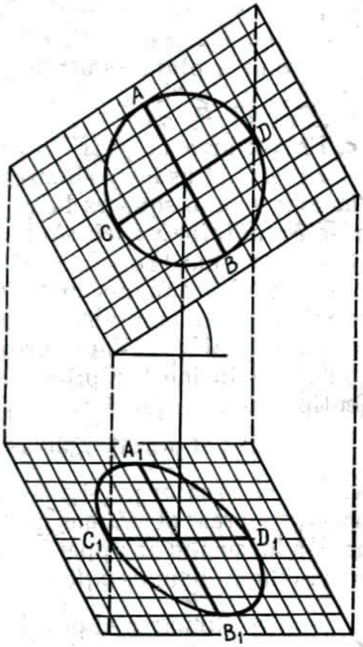


Fig. 106

$$\text{area of ellipse} = \pi R^2\cdot\cos\varphi = \pi R\cdot R\cos\varphi = \pi ab.$$

Thus, the area of an ellipse is equal to the product of its semi-axes multiplied by the number π.

Sec. 43. Polyhedral Angles

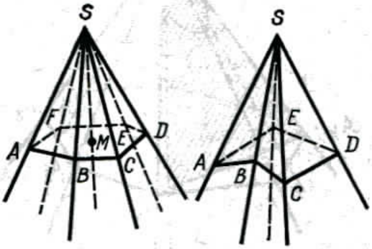
Let $ABC\ \ldots\ F$ be a polygon and S a point lying outside the plane containing the polygon (Fig. 107,*a*).

Definition. A set of all the points belonging to the rays [SM) where M runs along the polygon $ABC\ \ldots\ F$ is called a *polyhedral angle*.

The point S is called the *vertex* of a polyhedral angle, the rays $[SA)$, $[SB)$, $[SC)$ are the *edges* of a polyhedral angle, and the angles ASB, BSC, . . . are its *faces* (or *face angles*).

The faces of a polyhedral angle form its bound (or surface). The points of a polyhedral angle not belonging to its surface form the interior domain of the polyhedral angle.

A polyhedral angle will be *convex* if its interior domain lies entirely on one side of any plane containing one of its faces. Fig. 107 illustrates a convex (*a*) and a concave (*b*) polyhedral angles. In our course we shall consider only convex angles.



(a) (b)

Fig. 107

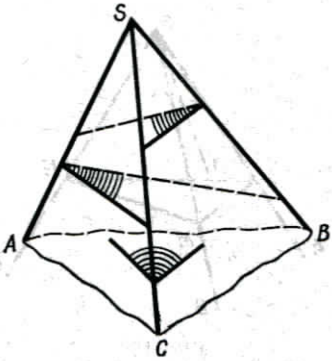


Fig. 108

Any two faces of a polyhedral angle having a common edge form a dihedral angle. Plane angles of these dihedral angles are shown in Fig. 108.

Polyhedral angles are also classified according to the number of their faces (or edges) as trihedral, tetrahedral, pentahedral, etc.

Let us consider the following properties of polyhedral angles.

Theorem 1. *The magnitude of any face angle of a convex polyhedral angle is less than the sum of magnitudes of the remaining angles.*

Given: a polyhedral angle $SA_1A_2A_3 \ldots A_n$; and the magnitudes of its face angles $\varphi_1, \varphi_2, \ldots, \varphi_n$.

Prove: $\varphi_i < \varphi_1 + \varphi_2 + \ldots + \varphi_{i-1} + \varphi_{i+1} + \ldots + \varphi_n$ for $\forall\, i \leqslant n$.

Proof. 1. Let us first show that the magnitude of any face angle of a trihedral angle is less than the sum of the other two of its face angles.

Suppose that in the trihedral angle $SABC \angle ASC$ is the largest face angle (Fig. 109) and draw a ray $[SD)$ inside it so that $\angle CSD$ is congruent to $\angle CSB$. We then lay off congruent line segments $[SD]$ and $[SM]$ on the edges $[SD)$ and $[SB)$, and draw an arbitrary plane (not containing S) through the points D and M. This plane will intersect the edge $[SA)$ at some point L, and the edge $[SC)$ at a point N (Fig. 109). The triangles DSN and MSN are congruent, since $[SN]$ is a common side, $|SD| = |SM|$, and the angle DSN is congruent to the angle MSN. Hence, $|DN| = |MN|$.

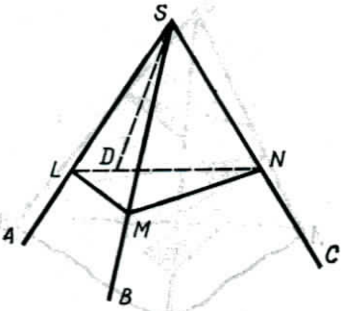


Fig. 109

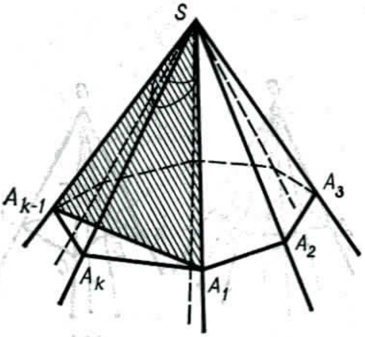


Fig. 110

From the triangle LMN we have $|LN| < |LM| + |MN|$, or $|LD| + |DN| < |LM| + |MN|$. Since $|DN| = |MN|$, we get $|LD| < |LM|$. Comparing now the triangles LSD and LSM, where $[SL]$ is a common side, $[SD] \cong [SM]$ and $|LD| < |LM|$, we conclude that $\widehat{LSD} < \widehat{LSM}$.

Adding $\widehat{DSN}$ to the left-hand member of the last inequality and $\widehat{MSN}$ (which is congruent to $\widehat{DSN}$) to the right-hand member, we get $\widehat{LSD} + \widehat{DSN} < \widehat{LSN} + \widehat{MSN}$, or

$$\widehat{ASC} < \widehat{ASB} + \widehat{BSC}.$$

2. We shall prove the general case using the method of induction. Assume that the theorem is true for $n = k -$

-1, i.e.

$$\varphi_i < \varphi_1 + \varphi_2 + \dots + \varphi_{i-1} + \varphi_{i+1} + \dots + \varphi_{k-1}.$$

Let us show that it is true for $n = k$, i.e.

$$\varphi_i < \varphi_1 + \varphi_2 + \dots + \varphi_{i-1} + \varphi_{i+1} + \dots + \varphi_k.$$

Through the edges $[SA_1)$ and $[SA_{k-1})$ we draw a plane cutting off the given polyhedral angle a trihedral angle $SA_1A_kA_{k-1}$, in which $\widehat{A_{k-1}SA_1} < \varphi_{k-1} + \varphi_k$ (Fig. 110). By the induction assumption, $\varphi_i < \varphi_1 + \varphi_2 + \dots + \varphi_{i-1} + \varphi_{i+1} + \dots + A_1SA_{k-1}$. Comparing both inequalities, we get

$$\varphi_i < \varphi_1 + \varphi_2 + \dots + \varphi_{i-1} + \varphi_{i+1} + \dots + \varphi_{k-1} + \varphi_k,$$

which was required to be proved.

Theorem 2. *The sum of the face angles of a convex polyhedral angle is less than* 360°.

Given: a convex polyhedral angle $SABCDE$.

Prove: $\Sigma < 360°$, where Σ is the sum of face angles.

Proof. Let us draw a plane α not passing through the vertex S and intersecting all the faces of the polyhedral angle $SABCDE$ (Fig. 111). Each vertex of the polygon $ABCDE$ is the vertex of the trihedral angle. By virtue of Theorem 1, we have for these angles

$$\widehat{ABC} < \widehat{ABS} + \widehat{SBC},$$

$$\widehat{BCD} < \widehat{BCS} + \widehat{SCD},$$

$$\widehat{CDE} < \widehat{CDS} + \widehat{SDE},$$

$$\widehat{DEA} < \widehat{DES} + \widehat{SEA}$$

$$\widehat{EAB} < \widehat{EAS} + \widehat{SAB}.$$

We then add both the left-hand and the right-hand sides of these inequalities. The left-hand member of the inequality obtained is the sum of the magnitudes of the interior angles of the convex polygon which is equal to $2dn - 4d$, where n is the number of the sides of the polygon. The right-hand member of the obtained inequality is the sum of the magnitudes of the interior angles of the triangles less the sum of the magnitudes of all the angles of the triangles which have the vertex at point S, i.e. $2dn - \Sigma$.

Thus, $2dn - 4d < 2dn - \sum$, whence it follows that $\sum < 360°$.

Example. In the trihedral angle depicted in Fig. 112 the face angle BSC is a right angle, the two other angles ASB and ASC being equal to 60° each. The plane (ABC) cuts off three equal segments from the edges $[SA]$, $[SB]$, and $[SC]$. Find the dihedral angle $ABCS$.

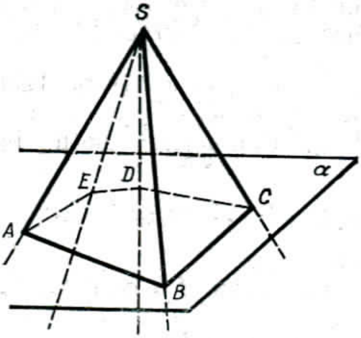


Fig. 111

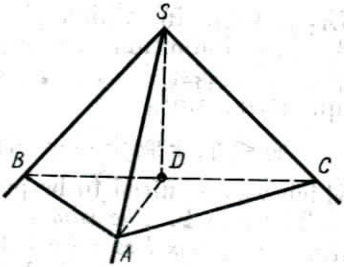


Fig. 112

Solution. Obviously, $|SA| = |SB| = |SC| = |AB| = |AC|$, therefore $\triangle BSC \cong \triangle ABC$ and $\widehat{BAC} = 90°$.

Let us construct the plane angle of the dihedral angle $SBCA$: from the vertex S we draw $[SD] \perp [BC]$ and join $D = (SD) \cap (BC)$ to A. Since $[AD]$ and $[SD]$ are perpendicular to $[BC]$, $\angle ADS$ is the plane angle of the required dihedral angle.

Denoting $|SA| = |SB| = |SC| = |AB| = |AC| = a$, we have $|SD| = |DA| = |DB|$ and $|SD|^2 + |BD|^2 = a^2$. Thus, $|DS|^2 + |AD|^2 = |AS|^2$, which means (see Sec. 42) that $\widehat{ADS} = 90°$.

Sec. 44. Worked Problems

Problem 1. The hypotenuse of a right-angled triangle lies in a plane α, and the legs are inclined to it at angles γ and φ. Find the angle between the plane containing the triangle and the plane α.

Solution. In the triangle ABC we draw the altitude $[CF]$ perpendicular to $[AB]$, and project it on the plane α: $ADB = \mathrm{pr}_\alpha\, ACB$, $[DF] = \mathrm{pr}_\alpha\, [CF]$ (Fig. 113). By the theorem on three perpendiculars, $[DF]$ is perpendicular to $[AB]$, consequently, $\angle CFD$ is the plane angle of the required dihedral angle $CABD$. At the same time it is an acute angle of the right-angled triangle CFD.

Let us introduce an auxiliary linear element $|CD| = h$. Then

$$|AC| = \frac{h}{\sin\gamma} \text{ (from } \triangle ACD), \qquad |BC| = \frac{h}{\sin\varphi} \text{ (from } \triangle BCD).$$

$$|AB| = \frac{h}{\sin\gamma \sin\varphi}\sqrt{\sin^2\gamma + \sin^2\varphi}$$

Since $|AB|\cdot|CF| = |AC|\cdot|BC|$, we have

$$|CF| = \frac{|AC|\cdot|BC|}{|AB|} = \frac{h^2 \sin\gamma \sin\varphi}{\sin\gamma \sin\varphi \sqrt{\sin^2\gamma + \sin^2\varphi}} = \frac{h}{\sqrt{\sin^2\gamma + \sin^2\varphi}}.$$

Thus, in the triangle CFD the leg $|CD| = h$, the hypotenuse $|CF| = h/\sqrt{\sin^2\gamma + \sin^2\varphi}$.
Hence,

$$\sin \widehat{CFD} = |CD|/|CF| = \sqrt{\sin^2\gamma + \sin^2\varphi} = \sqrt{\frac{1}{2}(1-\cos 2\gamma) + \frac{1}{2}(1-\cos 2\varphi)} = \sqrt{1-\cos(\gamma+\varphi)\cos(\gamma-\varphi)}.$$

From this equation it is seen that for any $0 < \gamma < 90°$ and $0 < \varphi < 90°$ the radicand is positive; hence, for these conditions there exists a solution which is unique. In particular, if $\varphi + \gamma = 90°$, then $\cos(\varphi+\gamma) = 0$ and $\sin\angle CFD = 1$. This means that the plane containing the triangle is perpendicular to the plane α.

Problem 2. The base of a triangular pyramid is an isosceles right-angled triangle ABC ($\hat{C} = 90°$). The length of the lateral edges of the pyramid is equal to b, and the length of the hypotenuse is c. Find the angles formed by the lateral edges with the base, and the dihedral angle at the edge $[SC]$ (Fig. 114).

Solution. Since the lateral edges have equal lengths, the angles formed by them with the plane containing the base are also equal. The altitude of the pyramid is then projected into the centre of the circle circumscribed about the base,

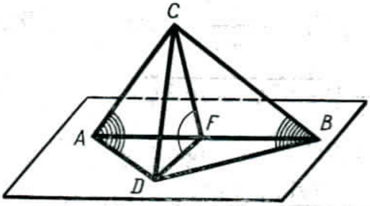


Fig. 113

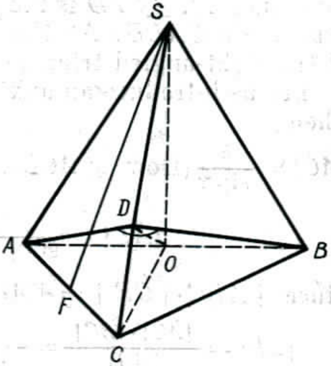


Fig. 114

i.e. into the midpoint of the hypotenuse of the right-angled triangle ABC. Thus, $[SO]$ is perpendicular to (ABC), and $\widehat{SAO} = \widehat{SBO} = \widehat{SCO}$.

To find these angles let us consider the triangle ASO, in which $|AS| = b$, $|AO| = c/2$. Then $\cos\widehat{OAS} = \cos\widehat{SBO} = \cos\widehat{SCO} = c/(2b)$.

To construct the plane angle of the dihedral angle $ACSB$ we draw $[AD]$ perpendicular to $[CS]$ and join the points D and B (Fig. 114). Since the triangles ASC and BSC are congruent, BD is perpendicular to $[SC]$ and, consequently, $\angle ADB$ is the required plane angle. Furthermore, $\triangle ADB$ is an isosceles triangle, $[DO]$ being its median, and, hence, its altitude, and its bisector. Therefore, $\widehat{ADB} = 2\widehat{ADO}$. In the triangle ADO we have $|AO| = c/2$, and the side $[AD]$ is found from the triangle ACS, in which $|SC| \times |AD| = |AC| \cdot |SF|$ $([SF] \perp [AC])$, where $|AC| = c/\sqrt{2}$. Thus

$$|AD| = \frac{|AC|\cdot|SF|}{|SC|} = \frac{c\sqrt{b^2 - \frac{c^2}{8}}}{\sqrt{2}\,b} = \frac{c\sqrt{8b^2 - c^2}}{4b}.$$

We now find

$$\sin \widehat{ADO} = \frac{2b}{\sqrt{8b^2-c^2}}; \quad ADB = 2 \arcsin \frac{2b}{\sqrt{8b^2-c^2}}.$$

Problem 3. In a cube the length of whose edge is equal to a find the distance between skew diagonals of two adjacent faces.

Solution. We first find the distance between the diagonals $[AD_1]$ and $[DC_1]$ (Fig. 115). This distance is equal to the distance from any point of one line, say (AD_1) to the plane passing through the other line parallel to the first one (see Sec. 36). Since (AD_1) is parallel to (C_1B), (AD_1) is parallel to (DC_1B).

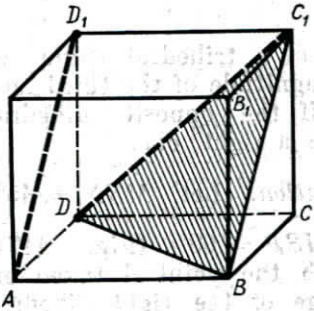


Fig. 115

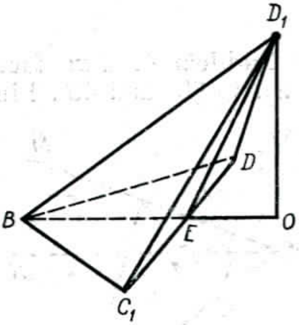


Fig. 116

Let us now find the distance from the point $D_1 \in (AD_1)$ to the plane (DC_1B). From the point D_1 we drop a perpendicular $[D_1O]$ onto the plane (DC_1B). Joining the vertices D, B, and C_1 to the point D_1 we get a pyramid D_1DBC_1 in which $[D_1O]$ is the altitude (Fig. 116). The base of the pyramid is a regular triangle DC_1B ($|DB| = |BC_1| = |DC_1| = a\sqrt{2}$), $|D_1C_1| = |D_1D| = a$, and $|D_1B| = a\sqrt{3}$. Since $|D_1C_1| = |D_1D|$, the foot of the altitude lies on the bisector of the angle B.

Consider the triangle BD_1E, in which $[D_1O]$ is the altitude. In this triangle all the sides are known: $|BE| = a\sqrt{6}/2$, $|D_1E| = a\sqrt{2}/2$, $|D_1B| = a\sqrt{3}$.

Let us note that $|D_1B|^2 > |BE|^2 + |D_1E|^2$, i.e. D_1BE is an obtuse-angled triangle (see Fig. 116) and determine $\widehat{D_1BE}$:

$$\cos D_1\widehat{B}E = \frac{|D_1B|^2+|BE|^2-|ED_1|^2}{2|D_1B|\cdot|BE|} =$$

$$= \frac{3a^2+3a^2/2-a^2/2}{2a\sqrt{3}\cdot a\sqrt{6}/2} = \frac{4}{3\sqrt{2}}.$$

Hence,

$$|OD_1| = \sin D_1\widehat{B}E\cdot|D_1B| = a\sqrt{3}\sqrt{1-\cos^2 D_1\widehat{B}E} = a\frac{\sqrt{3}}{3}.$$

Problem 4. Two face angles of a trihedral angle are equal to 60° and 45°. Find the magnitude of the third face angle if the opposite dihedral angle is a right one.

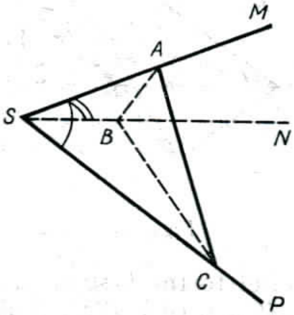


Fig. 117

Solution. Let $\widehat{MSN} = 45°$, and $\widehat{MSP} = 60°$ (Fig. 117). Through the point A taken on the edge of the right dihedral angle we draw a plane perpendicular to this edge which intersects the faces of the trihedral angle along the line segments $[AC]\perp[SA]$, $[AB]\perp[SA]$, and $[BC]$.

By hypothesis, $\widehat{BAC} = 90°$, $\widehat{ASB} = 45°$, and we have to determine the angle BSC. We shall find this angle from the triangle BSC, provided all its sides are expressed in terms of $|SA|$.

Putting $|SA| = a$ and considering $\triangle BSA$, we have $|AB| = |SA| = a$, $|SB| = a\sqrt{2}$. From $\triangle CSA$ we find $|AC| = |AS|\cdot\tan 60° = a\sqrt{3}$, $|SC| = |AS| : \cos 60° = 2a$. Hence, $|BC| = \sqrt{|AB|^2+|AC|^2} = 2a$. Returning to $\triangle BSC$, we get

$$\cos B\widehat{S}C = \frac{|SB|^2+|SC|^2-|BC|^2}{2|BS|\cdot|SC|} = \frac{2a^2+4a^2-4a^2}{2a\sqrt{2}\cdot 2a} = \frac{\sqrt{2}}{4}.$$

PROBLEMS TO CHAPTER 5

1. From a point A 12 cm distant from a plane an inclined line $[AB]$ of length 37 cm is drawn to this plane. Find the length of the projection of the line $[AB]$ on the given plane.

2. From the centre of a circle whose radius is equal to 18 cm a perpendicular is erected to the plane of the circle. Find the distance from the end-point of this perpendicular to the circle if the perpendicular is 80 cm long.

3. From the centre O of a circle whose radius is equal to 3 dm a perpendicular $[OB]$ is erected to its plane. $[AC]$ is a segment of a line tangent to the circle at point A. Find $|BC|$ if $|AC| = 2$ dm and $|OB| = 6$ dm.

4. In a rectangle $ABCD$ $|AB| = 9$ cm, $|BC| = 8$ cm; $[DF]$ is perpendicular to $(ABCD)$. Find the distance from F to the vertices of the rectangle if $|DF| = 12$ cm.

5. From a point situated outside a plane an inclined line 20 cm long is drawn to this plane at an angle of 45°. Find the distance from the given point to the plane.

6. Find the distance from the point A to a plane α if the distances from A to the points B and C lying in the plane α are equal to 51 cm and 30 cm, respectively, and the lengths of the projections of the inclined lines $[AB]$ and $[AC]$ are as $5:2$.

7. Drawn from a point located outside a plane are a perpendicular of length 12 cm, and an inclined line (16 cm long) to this plane. Find the length of the projection of the perpendicular on the inclined line.

8. The legs of a right-angled triangle ABC are equal to 12 dm and 16 dm. From the vertex C of the right angle a perpendicular $[CM]$ whose length is equal to 28 dm is erected to the plane. Find the distance from the point M to the hypotenuse.

9. The sides of a triangle are 15 cm, 37 cm, and 44 cm. From the vertex of the largest angle of the triangle a perpendicular is erected to the plane of the triangle equal to 16 cm. Find the distances from its end-points to the longest side.

10. From a point D lying outside a plane three inclined lines are drawn, each of which forms an angle of 60° with the given plane. The feet of the inclined lines A, B, and C are joined with line segments. Find the sides of the triangle ABC if the distance from the point D to the plane is equal to a and the angles ADB, BDC, CDA are congruent.

11. From a line segment $[AB]$ found outside a plane α two perpendiculars are dropped onto this plane ($[AC]$ and $[BD]$). Find the length of the projection of the line segment $[AB]$ on this plane if $[AB] = 26$ cm, $|AC| = 32$ cm, and $|BD| = 22$ cm.

12. From the end-points of a line segment $[AB]$ not lying in a plane α two perpendiculars are dropped onto this plane: $|AC| = 80$ cm and $|BD| = 60$ cm. Find the distance from the mid-point of the line segment $[AB]$ to the same plane.

13. The line segment $[AB]$ is parallel to a plane α; $B_1 = \mathrm{pr}_\alpha B$. Find the length of the line segment $[AB_1]$ if it forms an angle of 30° with the plane α.

14. From a point M on a straight line (AB) which is parallel to a plane α and 0.5 m distant from it two lines of equal length are drawn inclined to this plane and perpendicular to (AB). Find the length of the inclined lines if the angle between them is equal to 120°.

15. From a point M situated outside two parallel planes two straight lines are drawn intersecting the planes at points A, B, A_1, and B_1, respectively. Find the length of the line segment $[AA_1]$ if $|BB_1| = 28$ cm, and $|MA| : |AB| = 5 : 2$.

16. The base $[AB]$ of a trapezium $ABCD$ lies in a plane α and the base $[CD]$ is 4 dm distant from this plane. At what distance from the plane α is the point of intersection of the diagonals of the trapezium if $|AB| : |DC| = 5 : 3$?

17. One of the sides of a rhombus lies in a plane α, while the opposite side is 16 cm distant from this plane. The projections of the diagonals of the rhombus on the plane α are equal to 32 cm and 8 cm. Find the length of the side of the rhombus.

18. Two line segments are contained between two parallel planes. The projections of these segments on the planes are equal to 1 dm and 7 dm. Find the lengths of the segments if the difference between them is equal to 4 dm.

19. A triangle is located outside a plane α. The distances from its vertices to the plane are equal to 4 dm, 5 dm, and 9 dm. Find the distance from the centroid of this triangle to the plane α.

20. Through the vertex C of the right angle of a triangle a plane is drawn parallel to the hypotenuse at a distance of 20 cm from it. The projections of the legs on this plane are 60 cm and 100 cm long. Find the length of the hypotenuse.

21. One leg of an isosceles right-angled triangle lies in a plane α. The hypotenuse forms an angle of 30° with this plane. At what angle is the other leg inclined to the plane α?

22. A point M is 15 cm distant from a plane α. Find the distance from M to a straight line p contained in α if $\mathrm{pr}_{\alpha} M$ is 20 cm distant from p.

23. Given are two mutually perpendicular planes and a straight line intersecting them. The segment of this line enclosed between the planes is equal to l, its projections on the given planes being equal to p and q. Find the projection of the line segment on the line of intersection of the planes.

24. The distance from the point M taken inside a dihedral angle to either of its faces is equal to 2 dm. Find the distance of the point M from the edge of the dihedral angle if the angle between the perpendiculars dropped from M onto its faces is equal to 120°.

25. A point is taken inside a right dihedral angle at distances of 12 cm and 16 cm from its faces. Find the distance of this point from its edge.

26. A dihedral angle is equal to 45°. The distance from a point belonging to one face to the other face is equal to 1.5 dm. Find the distance of this point from the edge.

27. A dihedral angle is equal to 120°. A point taken inside it is 6 dm equidistant from its faces. Find the distance between the feet of the perpendiculars dropped from the point onto each face.

28. The distance of the point M lying on one face of a dihedral angle from the other face is equal to 12 cm. At the point M a perpendicular is erected to the face containing this point which cuts the other face. Find the length of this perpendicular if the dihedral angle is equal to 60°.

29. Find the length of the segment $[AB]$ contained between the faces of a dihedral angle if the projections of $[AB]$ on the faces are equal to 25 cm and 21 cm, and its projection on the edge is equal to 15 cm.

30. Through the hypotenuse of a right-angled triangle a plane is drawn which forms an angle of 60° with the plane containing the triangle. Find the distance from the vertex of the right angle of the triangle to this plane if the legs are equal to 6 dm and 8 dm.

31. The hypotenuse of an isosceles right-angled triangle lies in a plane α and is equal to 40 cm. The plane is inclined to the plane containing the triangle at an angle of 30°. Find the distance from the vertex of the right angle to the plane α.

32. From a point on the edge of a right dihedral angle two straight lines are drawn on the faces at an angle of 45° to the edge. Find the angle between the lines.

33. On a face of a dihedral angle two points are taken, which are 10 cm and 16 cm distant from its other face. Find the distance of the first point from the edge of the dihedral angle if the second point is 32 cm distant from the edge.

34. The end-points of a line segment 7.5 dm long lie on the faces of a right dihedral angle and are 50 cm and 55 cm distant from the edge. Find the length of the projection of the line segment on the edge of the dihedral angle.

35. How many faces has a convex polyhedral angle if each of its face angles is equal to: (1) 90°; (2) 60°; (3) 45°; (4) 30°?

36. Each face angle of a trihedral angle is equal to 60°. On one of the edges a line segment 9 dm long is laid off from the vertex, and from its end-point a perpendicular is drawn to the opposite face. Find (to within 0.1 cm) the length of this perpendicular.

37. In a trihedral angle two face angles are equal to 60° (each). On their common edge a line segment 4 dm long is laid off from the vertex. Find its projection on the plane containing the third face angle which is equal to 90°.

38. Inside a trihedral angle with right face angles a point is situated at distances a, b, and c from its edges. Find the distance of this point from the vertex of the trihedral angle.

39. From the vertex of a trihedral angle with right face angles a line segment is drawn inside the angle whose projections on the edges of the trihedral angle are equal to 25 cm, 50 cm, and 50 cm. Find the length of this line segment.

40. Prove that straight lines p and q are mutually perpendicular if p is parallel to a plane α, and q is perpendicular to it.

41. Through three points lying on the edges of a trihedral angle with right face angles a plane is drawn intersecting the edges at points A, B, and C. Prove that the square of the area of the triangle ABC is equal to the sum of the squares of the areas of the triangles AOB, BOC, and AOC, where O is the vertex of the trihedral angle.

42. Through the edge of the base of a cube a plane is drawn at an angle of 28° to its base. Find the area of the section figure if the edge of the cube is equal to 31 cm.

CHAPTER 6

THE COORDINATE METHOD

Sec. 45. Rectangular Coordinate System

Let us first consider two mutually perpendicular straight lines on a plane intersecting at point O. From this point we lay off two unit vectors on these lines (see Fig. 118). One of the lines (which are termed coordinate axes) is usually called the *axis of abscissas* (or the horizontal axis); the unit vector laid off on it is denoted by $\vec{i}$. On the other line called the *axis of ordinates* (or the vertical axis) the unit vector $\vec{j}$ is laid off on that side of the point O which enables the least rotation from $\vec{i}$ through $\vec{j}$ to be carried out anticlockwise. Thus, the unit vectors $\vec{i}$ and $\vec{j}$ specify directions and scale units on the coordinate axes.

Definition 1. A combination of two mutually perpendicular straight lines with established directions and an appropriate scale unit is called a *rectangular coordinate system on a plane.*

Let $\vec{a}$ be an arbitrary vector lying in a plane α on which a rectangular coordinate system is specified. Since the vectors $\vec{i}$, $\vec{j}$, and $\vec{a}$ are coplanar, then (see Theorem 2 of Sec. 27) the vector $\vec{a}$ can be uniquely presented in the form $\vec{a} = x\vec{i} + y\vec{j}$. To obtain this decomposition, the vector $\vec{a}$ must be translated so that its origin is brought into coincidence with the origin of the coordinate system, i.e. with the point O. Then $\vec{a}$ is the diagonal of a rectangle constructed on the vectors $x\vec{i}$ and $y\vec{j}$ as sides (see Fig. 118). The vectors $x\vec{i}$ and $y\vec{j}$ are called the *components* of the vector $\vec{a}$, and the numbers x and y its *coordinates* in the given coordinate system, x being the *abscissa* and y the *ordinate*. The coordinates of a vector are written in braces: $\vec{a}\ \{x,\ y\}$ or $\vec{a} = \{x,\ y\}$.

Let M be an arbitrary point in a plane. The vector $\overrightarrow{OM}$ drawn from the origin O of the coordinate system to the point M is termed the *radius vector* of the point M (Fig. 119).

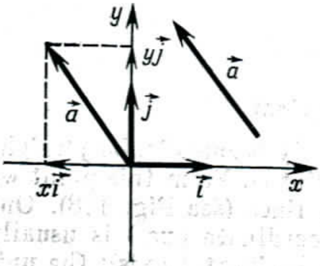


Fig. 118

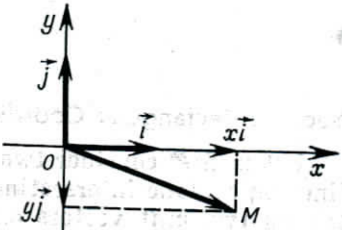


Fig. 119

Definition 2. The *coordinates of a point M* are defined as the coordinates of its radius vector $\overrightarrow{OM}$: $M\ (x,\ y)$, or $M = (x,\ y)$.

Let us take three mutually perpendicular straight lines in space which intersect at point O, and lay off on them (from the point O) unit vectors denoted by $\vec{i}$, $\vec{j}$, and $\vec{k}$. Depending on how we lay off the vectors $\vec{i}, \vec{j}, \vec{k}$, we get a right-hand or a left-hand coordinate system.

A coordinate system is said to be *right-hand* if a rotation of the first vector which brings it to coincidence (by the shortest route) with the second vector is performed in an anticlockwise sense for an observer at the terminal point of the third vector (Fig. 120). If the rotation is clockwise (Fig. 121), then the coordinate system is called *left-hand*. We shall consider here the right-handed coordinate system only.

The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ specify the direction and the scale unit on each of the three straight lines, i.e. they specify a rectangular coordinate system in space (see Definition 1).

The axis containing the vector $\vec{i}$ is called the *axis of abscissas*, the axis containing the vector $\vec{j}$ the *axis of ordinates*, and the axis containing the vector $\vec{k}$ the *axis of z-coordinates.*

Let $\vec{a}$ be an arbitrary vector in space. According to Theorem 3 of Sec. 27, $\vec{a}$ is a linear combination of three non-coplanar vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, i.e. it can be expressed in the form $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, the numbers x, y, and z being defined uniquely.

To obtain this decomposition the vector $\vec{a}$ should be translated so that its origin coincides with the origin of the

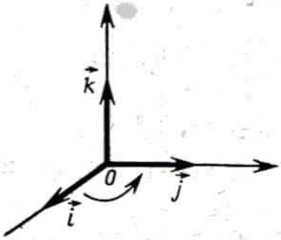


Fig. 120

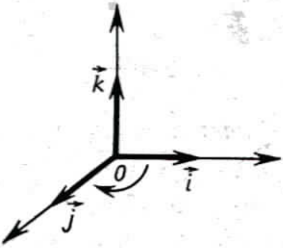


Fig. 121

coordinate system, i.e. with the point O. Then the vector $\vec{a}$ will coincide with the diagonal of the parallelepiped constructed on the vectors $x\vec{i}$, $y\vec{j}$, and $z\vec{k}$ as sides. These three vectors are termed the *components* of the vector $\vec{a}$, the numbers x, y and z being its *coordinates* in the given coordinate system: x is the *abscissa*, y the *ordinate*, and z is the *z-coordinate* (Fig. 122). As in the case of the vector in plane, the coordinates of a space vector are written in braces: $\vec{a}\ \{x, y, z\}$, or $\vec{a} = \{x, y, z\}$.

The coordinates of the point M in space coincide with the coordinates of its radius vector $\overrightarrow{OM}$: $M(x, y, z)$, or $M = (x, y, z)$.

The basic significance of the coordinate method consists in that linear operations on vectors are reduced to corresponding linear operations on numbers, i.e. on the coordinates of these vectors.

Theorem. *In adding two vectors $\vec{a}$ and $\vec{b}$ their like coordinates are added, and when multiplying a vector $\vec{a}$ by any number α all its coordinates are multiplied by this number.*

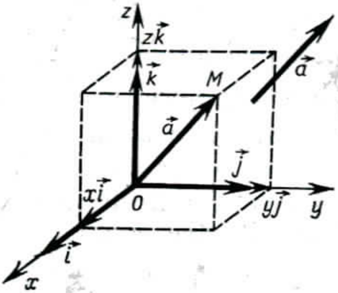


Fig. 122

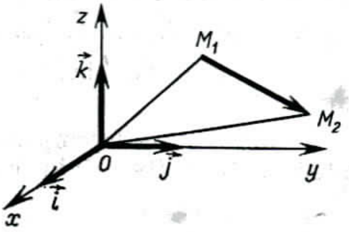


Fig. 123

Given: $\vec{a} = \{x_1, y_1, z_1\}$, $\vec{b} = \{x_2, y_2, z_2\}$.

Prove: $(\vec{a} + \vec{b}) = \{x_1 + x_2, y_1 + y_2, z_1 + z_2\}$, $\alpha\vec{a} = \{\alpha x_1, \alpha y_1, \alpha z_1\}$.

Proof. By hypothesis $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ and $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$. Then, using the property of the sum of vectors (see Sec. 25), we have

$$\vec{a}+\vec{b}=(x_1\vec{i}+y_1\vec{j}+z_1\vec{k})+(x_2\vec{i}+y_2\vec{j}+z_2\vec{k})=$$
$$=(x_1+x_2)\,\vec{i}+(y_1+y_2)\,\vec{j}+(z_1+z_2)\,\vec{k},$$

which means that the numbers $x_1 + x_2$, $y_1 + y_2$, $z_1 + z_2$ are the coordinates of the vector $\vec{a} + \vec{b}$, i.e. $(\vec{a} + \vec{b}) = \{x_1 + x_2, y_1 + y_2, z_1 + z_2\}$.

Analogously, by virtue of the property of multiplying a vector by a number (see Sec. 26), we have

$$\alpha\vec{a}=\alpha(x_1\vec{i}+y_1\vec{j}+z_1\vec{k})=\alpha x_1\vec{i}+\alpha y_1\vec{j}+\alpha z_1\vec{k}.$$

Hence the numbers αx_1, αy_1, αz_1 are the coordinates of the vector $\alpha\vec{a}$, i.e. $\alpha\vec{a} = \{\alpha x_1, \alpha y_1, \alpha z_1\}$.

Problem 1. Find the coordinates of the vector $\overrightarrow{M_1M_2}$, if the coordinates of its initial and terminal points are known: $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$.

Solution. We draw the radius vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ (Fig. 123). Obviously, $\overrightarrow{OM_2} = \overrightarrow{OM_1} + \overrightarrow{M_1M_2}$, i.e. $\overrightarrow{M_1M_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$. Since $\overrightarrow{OM_1} = \{x_1, y_1, z_1\}$, $\overrightarrow{OM_2} = \{x_2, y_2, z_2\}$, then, by the above theorem, we have

$$\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}.$$

We have obtained an important result: *the coordinates of a vector are equal to the difference of like coordinates of its initial and terminal points.*

Problem 2. Find the length of the vector $\vec{a}\ \{x, y, z\}$.

Solution. By hypothesis, $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, and the quantity $|\vec{a}|$ is equal to the diagonal of a rectangular parallelepiped, whose sides are $|x\vec{i}| = |x|$, $|y\vec{j}| = |y|$ and $|z\vec{k}| = |z|$. Hence,

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}.$$

Thus, the *length of a vector is equal to a square root of the sum of its squared coordinates.*

Since the coordinates of the point $M(x, y, z)$ coincide with the coordinates of its radius vector $\overrightarrow{OM}\ \{x, y, z\}$ and $|\overrightarrow{OM}|$ is equal to the distance of the point M from the origin of coordinates the result obtained may be formulated as follows: *the distance of a point $M(x, y, z)$ from the origin of coordinates is equal to a square root of the sum of the squares of its coordinates* $\rho(O, M) = \sqrt{x^2 + y^2 + z^2}$.

Problem 3. Find the distance between two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$.

Solution. Since $\rho(M_1, M_2) = |\overrightarrow{M_1M_2}|$, and $\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}$, then

$$\rho(M_1, M_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Sec. 46. Expressing a Scalar Product of Vectors in Terms of Their Coordinates. The Equation of a Plane

Let $\vec{a} = \{x_1, y_1, z_1\}$ and $\vec{b} = \{x_2, y_2, z_2\}$. According to the properties of a scalar product (see Sec. 29), we have

$$\vec{a}\cdot\vec{b} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k})\cdot(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}) =$$

$$= x_1x_2(\vec{i}\cdot\vec{i}) + x_1y_2(\vec{i}\cdot\vec{j}) + x_1z_2(\vec{i}\cdot\vec{k}) + y_1x_2(\vec{j}\cdot\vec{i}) +$$

$$+ y_1y_2(\vec{j}\cdot\vec{j}) + y_1z_2(\vec{j}\cdot\vec{k}) + (z_1x_2)(\vec{k}\cdot\vec{i}) + (z_1y_2)(\vec{k}\cdot\vec{j}) + z_1z_2(\vec{k}\cdot\vec{k}).$$

Since the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are mutually orthogonal and $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$, we have

$$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = \vec{i}\cdot\vec{k} = \vec{k}\cdot\vec{i} = \vec{k}\cdot\vec{j} = \vec{j}\cdot\vec{k} = 0,$$

and

$$|\vec{i}\cdot\vec{i}| = |\vec{j}\cdot\vec{j}| = |\vec{k}\cdot\vec{k}| = 1.$$

Therefore

$$\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2 = 0.$$

Thus, *a scalar product of two vectors is equal to the sum of products of their like coordinates.*

Knowing the coordinates of the vectors $\vec{a}$ and $\vec{b}$, it is easy to compute the angle between them. Indeed,

$$\cos(\widehat{\vec{a}, \vec{b}}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}. \qquad (1)$$

In particular, if the vectors $\vec{a}\,\{x_1, y_1, z_1\}$ and $\vec{b}\,\{x_2, y_2, z_2\}$ are orthogonal, then $\vec{a}\cdot\vec{b} = 0$ and, consequently,

$$x_1x_2 + y_1y_2 + z_1z_2 = 0. \qquad (2)$$

The equation thus obtained may be considered as the test for orthogonality of two non-zero vectors.

Example 1. Given the vectors $\vec{a} = \{6, 2, 3\}$ and $\vec{b} = \{1, 1, \sqrt{2}\}$. Find the angle between them.

Solution. By formula (1) we find

$$\cos\left(\widehat{\vec{a}, \vec{b}}\right)=\frac{6\cdot 1+2\cdot 1+3\sqrt{2}}{\sqrt{36+4+9}\sqrt{1+1+2}}\;\frac{8+3\sqrt{2}}{14}.$$

Hence, $\left(\widehat{\vec{a}, \vec{b}}\right) = \arccos\frac{8+3\sqrt{2}}{14}$.

Example 2. For what values of α are the vectors $\vec{a} = \{2, \alpha, \alpha\}$ and $\vec{b} = \{1, -6, 4\alpha\}$ orthogonal?

Solution. The given vectors are orthogonal if $\vec{a}\cdot\vec{b} = 0$. Expressing this equality in terms of coordinates, we get an equation for determining α: $2 - 6\alpha + 4\alpha^2 = 0$, $\alpha_1 = 1$, $\alpha_2 = -1/2$.

Problem 1. Write the equation of a plane passing through a given point $M_0(x_0, y_0, z_0)$ perpendicular to a given direction determined by the vector $\vec{n} = \{A, B, C\}$.

Solution. Any vector perpendicular to a plane is called its *normal vector*. Obviously, there exists the unique plane passing through a given point perpendicular to a given direction.

It is required to find an equation connecting x, y, and z such that:

(1) the coordinates x, y, z of any point belonging to the plane satisfy this equation;

(2) the coordinates of a point not belonging to the plane do not satisfy this equation.

To derive the equation of the plane α we take on it an arbitrary (moving) point $M(x, y, z)$: $M \in \alpha$. Since the point $M_0(x_0, y_0, z_0)$ also belongs to α, $\overrightarrow{MM_0}$ lies in the plane α, and $\vec{n}$ is perpendicular to $\overrightarrow{MM_0}$.

Making use of the condition (2) of orthogonality of two vectors, we have $\vec{n}\cdot\overrightarrow{MM_0} = 0$, where $\vec{n} = \{A, B, C\}$ and $\overrightarrow{MM_0} = \{x - x_0, y - y_0, z - z_0\}$. Writing a scalar product in the coordinate form, we get the required equation of the plane:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \qquad (3)$$

Indeed, the coordinates of any point $M \in \alpha$ satisfy this equation. But if the point $M(x, y, z) \in \alpha$, then $\overrightarrow{MM_0}$ is not perpendicular to n and therefore $\overrightarrow{MM_0} \cdot \vec{n} \neq 0$, i.e. its coordinates do not nullify the left-hand member of the equation obtained.

Example 3. Find the equation of the plane passing through the point $M\,(1,\ 2,\ -1)$ perpendicular to the bisector of the angle xOy.

Solution. Let us take, for instance, the vector $\vec{n} = \{1,\ 1,\ 0\}$ as the normal vector of the required plane. Then we get

$$1\,(x-1) + 1\,(y-1) + 0\,(z+1) = 0, \text{ or } x + y = 2.$$

Example 4. Write the equations of the coordinate planes xOy, yOz, xOz.

Solution. The plane xOy passes through the point $(0,\ 0,\ 0)$ and is perpendicular to the vector $\vec{k} = \{0,\ 0,\ 1\}$, its equation has the form $0\,(x-0) + 0\,(y-0) + 1\,(z-0) = 0$, i.e. $z = 0$. The equations of the planes yOz and xOz ($x = 0$ and $y = 0$, respectively) are obtained in a similar way.

Problem 2. Write the equation of a plane passing through three given non-collinear points $M_0\,(x_0,\ y_0,\ z_0)$, $M_1\,(x_1,\ y_1,\ z_1)$, and $M_2\,(x_2,\ y_2,\ z_2)$.

Solution. Let us denote the required plane by α, and its normal vector by $\vec{n} = \{a,\ b,\ c\}$. Since the required plane passes through the point $M_0\,(x_0,\ y_0,\ z_0)$, its equation is written in the form

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0 \tag{4}$$

The points $M_1\,(x_1,\ y_1,\ z_1)$ and $M_2\,(x_2,\ y_2,\ z_2)$ lie on this plane which means that their coordinates satisfy the equation (4):

$$\left.\begin{aligned} a\,(x_1 - x_0) + b\,(y_1 - y_0) + c\,(z_1 - z_0) = 0 \\ a\,(x_2 - x_0) + b\,(y_2 - y_0) + c\,(z_2 - z_0) = 0 \end{aligned}\right\}$$

Thus, we get a system of two homogeneous equations in three unknowns a, b, and c, from which we can express any two in terms of the third unknown which is non-zero.

For instance, we can find a and b ($c \neq 0$):

$$a = \frac{(y_1 - y_0)(z_2 - z_0) - (y_2 - y_0)(z_1 - z_0)}{(x_1 - y_0)(z_2 - z_0) - (y_2 - x_0)(z_1 - z_0)}\,c;$$

$$b = \frac{(x_1 - x_0)(z_2 - z_0) - (x_2 - x_0)(z_1 - z_0)}{(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)}\,c;$$

Substituting a and b into (4), we have

$$\begin{aligned}&[(y_1 - y_0)(z_2 - z_0) - (y_2 - y_0)(z_1 - z_0)]\, c\, (x_1 - x_0) + \\ &+ [(x_1 - x_0)(z_2 - z_0) - (x_2 - x_0)(z_1 - z_0)]\, c\, (y - y_1) + \\ &+ [(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)]\, c\, (z - z_0) = 0,\end{aligned}$$

and reducing by $c \neq 0$, we find the required equation:

$$\begin{aligned}&[(y_1 - y_0)(z_2 - z_0) - (y_2 - y_0)(z_1 - z_0)](x - x_0) + \\ &\quad + [(x_1 - x_0)(z_2 - z_0) - (x_2 - x_0)(z_1 - z_0)](y - y_0) + \\ &+ [(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)](z - z_0) = 0.\end{aligned} \tag{5}$$

It is easy to check that all three points M_0, M_1, and M_2 belong to this plane, i.e. their coordinates satisfy its equation.

Note. The equation obtained can be written in the form of a third-order determinant. The left-hand side of the equation (5) is an expansion of this determinant in terms of the elements of the first row.

The reader familiar with the vector product can derive the equation (5) in another way. Since the vectors $\overrightarrow{M_1M_0}$ and $\overrightarrow{M_2M_0}$ lie in the plane α, their vector product $\overrightarrow{M_1M_0} \times \overrightarrow{M_2M_0}$ is perpendicular to α and may be taken for a normal vector.

Sec. 47. Expressing the Vector Product of Two Vectors in Terms of Their Coordinates

Let $\vec{a} = \{x_1, y_1, z_1\}$ and $\vec{b} = \{x_2, y_2, z_2\}$. Using the properties of a vector product (see Sec. 30), we have

$$\begin{aligned}\vec{a} \times \vec{b} &= (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) \times (x_2\vec{i} + y_2\vec{j} + z_2\vec{k}) = \\ &= x_1x_2(\vec{i} \times \vec{i}) + x_1y_2(\vec{i} \times \vec{j}) + x_1z_2(\vec{i} \times \vec{k}) + \\ &+ y_1x_2(\vec{j} \times \vec{i}) + y_1y_2(\vec{j} \times \vec{j}) + y_1z_2(\vec{j} \times \vec{k}) + \\ &+ z_1x_2(\vec{k} \times \vec{i}) + z_1y_2(\vec{k} \times \vec{j}) + z_1z_2(\vec{k} \times \vec{k}).\end{aligned} \tag{1}$$

Since the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are mutually orthogonal and $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$, we have

$$\vec{i}\times\vec{j}=\vec{k}, \vec{j}\times\vec{i}=-\vec{k}, \vec{j}\times\vec{k}=\vec{i}, \vec{k}\times\vec{j}=-\vec{i}, \vec{k}\times\vec{i}=\vec{j}.$$
$$\vec{i}\times\vec{k}=-\vec{j}, \vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0.$$

Substituting the found vector products of the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ in equation (1), we get

$$\vec{a}\times\vec{b}=x_1y_2\vec{k}-x_1z_2\vec{j}-y_1x_2\vec{k}+y_1z_2\vec{i}+x_2z_1\vec{j}-z_1y_2\vec{i}=$$
$$=(y_1z_2-z_1y_2)\,\vec{i}-(x_1z_2-x_2z_1)\,\vec{j}+(x_1y_2-x_2y_1)\,\vec{k}.$$

For a better memorizing of this formula we write the contents of each parenthesis in the form of a determinant:

$$y_1z_2-z_1y_2=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad x_1z_2-x_2z_1=\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix},$$
$$x_1y_2-x_2y_1=\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Then

$$\vec{a}\times\vec{b}=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i}-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j}+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k}.$$

This result is easily memorized if use is made of the following table (matrix):

$$\begin{Vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{Vmatrix},$$

in which the first row contains the coordinates of the first vector $(\vec{a})$, and the second row the coordinates of the second vector $(\vec{b})$. Then the coefficient of $\vec{i}$ is the determinant obtained by deleting the first column from this table, the coefficient of $\vec{j}$ is the determinant obtained by deleting the second column and taken with the minus sign, and the coefficient of $\vec{k}$ is the determinant obtained by deleting the third column.

Example 1. Find the vector product of the vectors $\vec{a} = \vec{i} + 2\vec{j}$ and $\vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$.

Solution. Let us compile a table from the coordinates of the given vectors:

$$\begin{Vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{Vmatrix}.$$

Then

$$\vec{a}\times\vec{b}=\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix}\vec{i}-\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}\vec{j}+\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}\vec{k}=4\vec{i}-2\vec{j}-7\vec{k}.$$

Using the vector product, we can compute the area of a parallelogram, of a triangle, and, hence, of any polygon.

Example 2. On the edges $[AB]$, $[BC]$, and $[DD_1]$ of a cube $ABCDA_1B_1C_1D_1$ points of F, Q, and H are taken. Find the area of the triangle FQH, if $|AF|:|FB|=1:2$, $|BQ|:|QC|=1:1$, $|DH|:|HD_1|=1:3$.

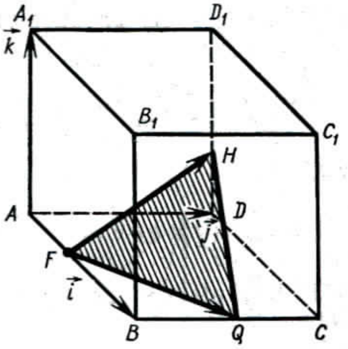


Fig. 124

Solution. The area of the triangle FQH is equal to half the area of the parallelogram constructed on $[FH]$ and $[FQ]$ as sides, and this area is numerically equal to the length of the vector product of the vectors $\vec{FH}$ and $\vec{FQ}$, i.e. $S_{FQH}=\frac{1}{2}\,|\vec{FH}\times\vec{FQ}|$.

Let us introduce a coordinate system so that $\vec{i}=\vec{AB}$, $\vec{j}=\vec{AD}$, $\vec{k}=\vec{AA_1}$ (Fig. 124). In this system $\vec{FH}=\vec{FA}+\vec{AD}+\vec{DH}=-\frac{1}{3}\vec{i}+\vec{j}+\frac{1}{4}\vec{k}=\left\{-\frac{1}{3},\ 1,\ \frac{1}{4}\right\}$, $\vec{QF}=\vec{FB}+\vec{BQ}=\frac{2}{3}\vec{i}+\frac{1}{2}\vec{j}=\left\{\frac{2}{3},\ \frac{1}{2},\ 0\right\}$. Using the coordinates of these vectors, we compile a table

$$\begin{Vmatrix} -1/3 & 1 & 1/4 \\ 2/3 & 1/2 & 0 \end{Vmatrix},$$

we find

$$\vec{FH}\times\vec{FQ}=\begin{vmatrix} 1 & 1/4 \\ 1/2 & 0 \end{vmatrix}\vec{i}-\begin{vmatrix} -1/3 & 1/4 \\ 2/3 & 0 \end{vmatrix}\vec{j}+\begin{vmatrix} -1/3 & 1 \\ 2/3 & 1/2 \end{vmatrix}\vec{k}=$$
$$=-\frac{1}{8}\vec{i}+\frac{1}{6}\vec{j}-\frac{5}{6}\vec{k}.$$

Hence,

$$S_{FQH}=\frac{1}{2}\sqrt{\frac{1}{64}+\frac{1}{36}+\frac{25}{36}}=\frac{5}{48}\sqrt{17}\text{ unit}^2.$$

Sec. 48. Expressing the Triple Product of Three Vectors in Terms of Their Coordinates

Let $\vec{a}=\{x_1, y_1, z_1\}$, $\vec{b}=\{x_2, y_2, z_2\}$, $\vec{c}=\{x_3, y_3, z_3\}$. As is known, the triple product of three vectors is a number equal to a scalar product of the vector $\vec{a}\times\vec{b}$ by the vector $\vec{c}$ (see Sec. 37). Since

$$\vec{a}\times\vec{b}=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i}-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j}+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k},$$

we get

$$\vec{a}\vec{b}\vec{c}=(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}x_3-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}y_3+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}z_3.$$

This formula can be written in the form of a third-order determinant:

$$\vec{a}\,\vec{b}\,\vec{c}=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{1}$$

(see Note in Sec. 46).

Example. Find the triple product of three vectors $\vec{a}\,\{1, 0, -1\}$, $\vec{b}\,\{0, 1, 2\}$, and $\vec{c}\,\{3, -1, 1\}$.

Solution. By formula (1) we have

$$\vec{a}\,\vec{b}\,\vec{c}=\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}3-\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}(-1)+\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}1=3+2+1=6.$$

Sec. 49. Worked Problems

Problem 1. Find the angle between the diagonal $[BD_1]$ of the rectangular parallelepiped $ABCDA_1B_1C_1D_1$ and the diagonal $[A_1D]$ of its face, if $|AD|=1$, $|DC|=3$, $|DD_1|=\sqrt{15}$.

Solution. Let us introduce a rectangular coordinate system with the origin at point D, and with the axis of abscissas (or the x-axis) directed along the straight line (DA , the y-axis along the line (DC), and the z-axis along the line (DD_1) (Fig. 125). Then $\overrightarrow{BD_1} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DD_1} = -\vec{i} - 3\vec{j} + \sqrt{15}\,\vec{k} = \{-1, -3, \sqrt{15}\}$, $\overrightarrow{DA_1} = \overrightarrow{DA} + \overrightarrow{AA_1} = \vec{i} + \sqrt{15}\vec{k} = \{1, 0, \sqrt{15}\}$. According to the formula (1) of Sec. 46, we get

$$\cos(\widehat{\overrightarrow{BD_1}, \overrightarrow{DA_1}}) = \frac{1(-1)+(-3)\cdot 0+\sqrt{15}\cdot\sqrt{15}}{\sqrt{1+9+15}\cdot\sqrt{1+0+15}} = \frac{7}{10}.$$

Consequently, the angle between the lines (A_1D) and (BD_1) is equal to arccos (7/10).

Problem 2. The projection of an isosceles triangle ABC ($|AC| = |BC|$) is a right-angled triangle ABC_1.

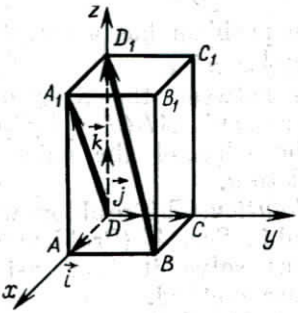


Fig. 125

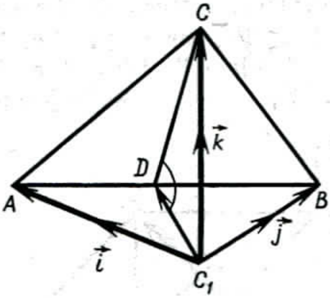


Fig. 126

Find the dihedral angle between the planes (ABC) and (ABC_1) if the altitude [CD] of the triangle ABC forms an angle φ with the side [C_1A] (Fig. 126).

Solution. Since [CC_1] is a perpendicular to the plane ABC_1, and [C_1D] is perpendicular to [AB], then $\angle CDC_1$ is the plane angle of the required dihedral angle. Suppose $|CC_1| = H$ and $\widehat{CDC_1} = \alpha$. Then $|C_1D| = H \cot \alpha$, $|C_1A| = |C_1B| = \sqrt{2}H \cot \alpha$.

We now introduce a rectangular coordinate system whose origin coincides with the point C_1, the x-axis with the line (C_1A), the y-axis with the line (C_1B), and the z-axis with the line (C_1C) (see Fig. 126).

Let us write in this system the coordinates of the vectors $\overrightarrow{CD}$ and $\overrightarrow{C_1A}$. Since the lengths of the vectors $\overrightarrow{C_1A}$, $\overrightarrow{C_1B}$, and $\overrightarrow{C_1C}$, lying on the coordinate axes are known, we have $\overrightarrow{c_1A} = \sqrt{2}H \cot \alpha \cdot \vec{i} = \{\sqrt{2}H \cot \alpha,\ 0,\ 0\}$, $\overrightarrow{C_1C} = H \cdot \vec{k} = \{0,\ 0,\ H\}$. Then $\overrightarrow{CD} = \overrightarrow{CC_1} + \overrightarrow{C_1D} = -\overrightarrow{C_1C} + \frac{1}{2}(\overrightarrow{C_1A} + \overrightarrow{C_1B}) = \left\{\frac{\sqrt{2}}{2} H \cot \alpha,\ \frac{\sqrt{2}}{2} H \cot \alpha,\ -H\right\}$.

By hypothesis, the angle between the vectors $\overrightarrow{C_1A}$ and $\overrightarrow{CD}$ is equal to φ, therefore, by the formula (1) of Sec. 46, we have

$$\cos \varphi = \frac{\overrightarrow{C_1A} \cdot \overrightarrow{CD}}{|\overrightarrow{C_1A}| \cdot |\overrightarrow{CD}|} = \frac{\sqrt{2}\, H \cot \alpha \frac{\sqrt{2}}{2} H \cot \alpha}{\sqrt{2}\, H \cot \alpha \cdot H \sqrt{1+\cot^2 \alpha}} = \frac{\cos \alpha}{\sqrt{2}},$$

whence $\alpha = \arccos (\sqrt{2} \cos \varphi)$. The problem has a solution if $0 < \sqrt{2} \cos \varphi < 1$, i.e. if $\pi/4 < \varphi < \pi/2$.

Problem 3. Find the distance between the diagonals $[AD_1]$, $[DC_1]$ of two adjacent faces of a cube $ABCDA_1B_1C_1D_1$ if the edge of the cube is equal to a.

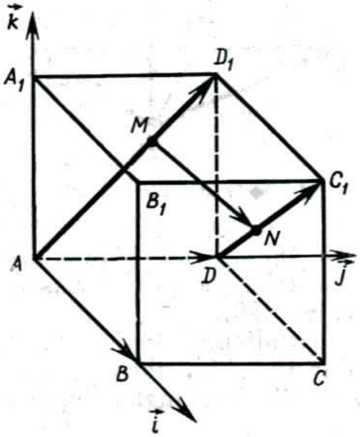


Fig. 127

Solution. This problem was solved in Sec. 44 (Problem 3). Let us solve it here using another method.

Let $[MN]$ be a common perpendicular to the lines (AD_1) and (DC_1), where $M \in (AD_1)$ and $N \in (DC_1)$ (Fig. 127). We then introduce a rectangular coordinate system, in which point A is the origin, the x-axis coincides with the line (AB), the y-axis with the line (AD), and the z-axis with the line (AA_1). In this system $\overrightarrow{AB} = a\vec{i}$, $\overrightarrow{AD} = a\vec{j}$, $\overrightarrow{AA_1} = a\vec{k}$, $\overrightarrow{AD_1} = \overrightarrow{AD} + \overrightarrow{DD_1} = \overrightarrow{AD} + \overrightarrow{AA_1} = a\vec{j} + a\vec{k} = \{0, a, a\}$, $\overrightarrow{DC_1} = \overrightarrow{DC} + \overrightarrow{CC_1} = \overrightarrow{AB} + \overrightarrow{AA_1} = a\vec{i} + a\vec{k} =$

$= \{a, 0, a\}$. Since $\overrightarrow{AM}$ is collinear with $\overrightarrow{AD_1}$ and $\overrightarrow{DN}$ is collinear with $\overrightarrow{DC_1}$, then $\overrightarrow{AM} = \alpha AD_1 = \{0, \alpha a, \alpha a\}$ and $\overrightarrow{DN} = \beta\overrightarrow{DC_1} = \{\beta a, 0, \beta a\}$, where α and β are numbers. Consequently, $\overrightarrow{MN} = \overrightarrow{AD} + \overrightarrow{DN} - \overrightarrow{AM} = \{\beta a, a \times \times (1-\alpha), (\beta-\alpha) a\}$. The numbers α and β can be found from the condition of perpendicularity of $\overrightarrow{MN}$ to the vectors $\overrightarrow{AD_1}$ and $\overrightarrow{DC_1}$ (see formula (2) of Sec. 46). We have

$$\left.\begin{aligned} a^2(1-\alpha)+a^2(\beta-\alpha)=0,\\ a^2\beta+a^2(\beta-\alpha)=0,\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned}\beta-\alpha+1=0,\\ 2\beta-\alpha=0\end{aligned}\right\}.$$

Solving this system, we find $\alpha = 2/3$, $\beta = 1/3$, and $\overrightarrow{MN} = \{a/3, a/3, -a/3\}$. The required distance $|MN| = \sqrt{\frac{a^2}{9}+\frac{a^2}{9}+\frac{a^2}{9}} = \frac{a\sqrt{3}}{3}$.

Prior to passing over to solving problems on computing section areas, let us derive one more very useful formula.

Problem 4. Prove that the area of any triangle ABC

$$S_{ABC} = \frac{1}{2}\sqrt{|AB|^2\cdot|BC|^2-(\overrightarrow{AB}\cdot\overrightarrow{BC})^2}. \tag{1}$$

Proof. We have

$$S_{ABC} = \frac{1}{2}|AB|\cdot|BC|\sin\varphi, \tag{2}$$

where $\varphi = \widehat{ABC}$. On the other hand,

$$\cos\varphi = \frac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{|AB|\cdot|BC|},$$

therefore

$$\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{1-\frac{(\overrightarrow{AB}\cdot\overrightarrow{BC})}{|AB|^2\cdot|BC|^2}} =$$

$$= \frac{1}{|AB|\cdot|BC|}\sqrt{|AB|^2|BC|^2-(\overrightarrow{AB}\cdot\overrightarrow{BC})^2}.$$

Substituting the found value of $\sin\varphi$ in (2), we get

$$S_{ABC} = \frac{1}{2}\sqrt{|AB|^2\cdot|BC|^2-(\overrightarrow{AB}\cdot\overrightarrow{BC})^2},$$

which was required to be proved.

Problem 5. A plane is drawn through the vertex A, the midpoint of the edge $[BC]$ and the centre of the face

CDC_1D_1. Find the area of the section thus obtained if the edge of the cube is equal to 1.

Solution. We introduce a coordinate system: $\vec{AB} = \vec{i}$, $\vec{AD} = \vec{j}$, $\vec{AA_1} = \vec{k}$, in which $A\ (0,\ 0,\ 0)$, $B\ (1,\ 0,\ 0)$, the midpoint of the edge $[BC]$, i.e. point $E\ (1,\ 1/2,\ 0)$, the centre of the face CDD_1C_1, i.e. the point $M\ (1/2,\ 1,\ 1/2)$ (Fig. 128).

We then find the intersection of the cutting plane with the edges $[DD_1]$ (point Q) and $[CC_1]$ (point F). Obviously $Q\ (0,\ 1,\ z_Q)$ and $F\ (1,\ 1,\ z_F)$, and we have to find z_Q and z_F.

The points Q and F lie in the plane passing through the given points $A\ (0,\ 0,\ 0)$, $E\ (1,\ 1/2,\ 0)$, $M\ (1/2,\ 1,\ 1/2)$.

Let us write the equation of this cutting plane.

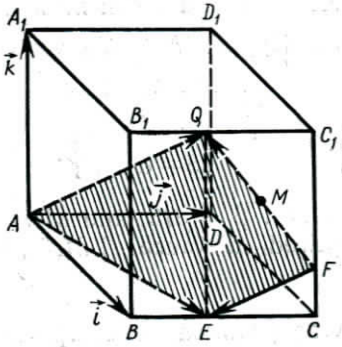


Fig. 128

Proceeding in the same way as in Problem 2 of Sec. 46, we introduce the coordinates a, b, c of the normal vector: $\vec{n} = \{a,\ b,\ c\}$.

Then

$$ax + by + cz = 0 \tag{3}$$

is the equation of the plane α.

The coordinates of the points E and M satisfy this equation (the point A was already used), i.e.

$$\left.\begin{aligned} a + \frac{1}{2}b &= 0 \\ \frac{1}{2}a + b + \frac{1}{2}c &= 0 \end{aligned}\right\}$$

Solving this system with respect to b and c, we find: $b = -2a$, $c = 3a$. Substituting these expressions into the equation (3) and reducing by $a \neq 0$, we find the equation of α: $x - 2y + 3z = 0$.

Now it is easy to obtain the coordinates of the points Q

and F. Since $Q \in \alpha$ and $F \in \alpha$, then

$$\frac{1}{4}\cdot 0-\frac{1}{2}\cdot 1+\frac{3}{4}z_Q=0 \text{ and } \frac{1}{4}\cdot 1-\frac{1}{2}\cdot 1+\frac{3}{4}z_F=0,$$

whence $z_Q = 2/3$, $z_1 = 1/3$, and $Q\ (0,\ 1,\ 2/3)$, $F\ (1,\ 1,\ 1/3)$.

Thus, the section figure is a quadrilateral $AQFE$, the coordinates of whose vertices are known. Its area is equal to the sum of the triangles QAE and QFE to be computed by the formula (1):

$$S_{AQFE}=S_{QAE}+S_{QFE}=\frac{1}{2}\sqrt{|AQ|^2\cdot|AE|^2-(\vec{AQ}\cdot\vec{AE})^2}+$$
$$+\frac{1}{2}\sqrt{|FQ|^2|FE|^2-(\vec{FQ}\cdot\vec{FE})^2}. \qquad (3)$$

Since $\vec{AQ}=\{0,\ 1,\ 2/3\}$, $\vec{AE}=\{1,\ 1/2,\ 0\}$, $\vec{FQ}=\{-1,\ 0,\ 1/3\}$, $\vec{FE}=\{0,\ -1/2,\ -1/3\}$,

$$|AQ|^2=1+\frac{4}{9}=\frac{13}{9},\quad |AE|^2=1+\frac{1}{4}=\frac{5}{4},$$
$$|FQ|^2=1+\frac{1}{9}=\frac{10}{9},$$
$$|FE|=\frac{1}{4}+\frac{1}{9}=\frac{13}{36},\ \vec{AQ}\cdot\vec{AE}=\frac{1}{2},\ \vec{FQ}\cdot\vec{FE}=\frac{1}{9},$$

then, substituting these values in the equation (3), we find

$$S_{\text{sec}}=\frac{1}{2}\sqrt{\frac{13}{9}\cdot\frac{5}{4}-\frac{1}{4}}+\frac{1}{2}\sqrt{\frac{10}{9}\cdot\frac{13}{36}-\frac{1}{81}}=$$
$$=\frac{1}{2}\left(\frac{\sqrt{14}}{3}+\frac{\sqrt{14}}{6}\right)=\frac{\sqrt{14}}{4}\text{ unit}^2.$$

Problem 6. A cube is cut with a plane passing through one of its diagonals; the length of its edge is equal to 1. How must this plane be drawn to obtain the least area of the section?

Solution. Suppose this plane passes through the diagonal $[AC_1]$ (Fig. 129) and intersects the edge $[BB_1]$ at point E (the case when the plane intersects the edge $[A_1D_1]$ or $[A_1B_1]$ is considered analogously and yields the same result). It is obvious, that the point E determines the section. Denoting the intersection of the cutting plane with the edge (DD_1) by F, we see that the section AEC_1F is a parallelogram, since $[EC_1] \parallel [AF]$ and $[C_1F] \parallel [AE]$.

We now introduce a coordinate system, putting $\vec{AB} = \vec{i}$, $\vec{AD} = \vec{j}$, $\vec{AA_1} = \vec{k}$, and denote $|BE|$ by z. The area

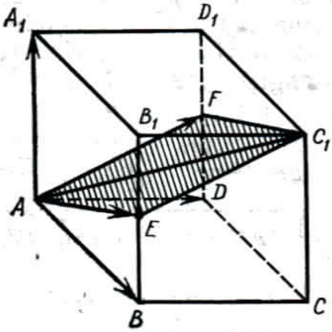


Fig. 129

of the parallelogram AEC_1F is equal to twice the area of the triangle AEC_1, therefore

$$S_{\text{sec}} = 2 \cdot S_{AEC_1} = \sqrt{|AE|^2 \cdot |AC_1|^2 - (\vec{AE} \cdot \vec{AC_1})^2}.$$

Since $\vec{AE} = \{1, 0, z\}$, $\vec{AC_1} = \{1, 1, 1\}$, we have $|AE|^2 = 1 + z^2$, $|AC_1|^2 = 3$, $\vec{AE} \cdot \vec{AC_1} = 1 + z$, and, hence,

$$S_{\text{sec}} = \sqrt{3(1+z^2) - (1+z)^2} = \sqrt{2(z^2 - z + 1)} \text{ unit}^2.$$

Let us find out for what values of z S_{sec} attains the least value, for which purpose we write it in the form

$$S_{\text{sec}} = \sqrt{2\left[\left(z - \frac{1}{2}\right)^2 + \frac{3}{4}\right]} \text{ unit}^2.$$

Whence it is seen that the least value of S is attained for $z = 1/2$, i.e. when the cutting plane passes through the midpoint of the edge $[BB_1]$, and

$$S_{\text{least}} = \sqrt{6}/2 \text{ unit}^2.$$

PROBLEMS TO CHAPTER 6

1. Given points $A\ (3, -1, 2)$ and $B\ (-1, 2, 1)$. Find the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

2. Compute the length of the vector $\vec{a} = \{6, 3, -2\}$.

3. Given two coordinates of a vector: $x = -1$, and $y = 2$. Find the third coordinate of the vector if its length is equal to $\sqrt{14}$.

4. The terminal point of the vector $\vec{a} = \{3, -1, 4\}$ coincides with the point N, and the initial point with the point $M\ (1, 2, -3)$. Find the coordinates of the point N.

5. Determine the coordinates of the initial point of the vector $\vec{a} = \{2, -3, -1\}$ if its terminus coincides with the point $(1, -1, 2)$.

6. The vectors $\overrightarrow{AB} = \{2, 6, -4\}$ and $\overrightarrow{AC} = \{4, 2, -2\}$ coincide with the sides of a triangle ABC. Determine the coordinates of the vectors reduced to the vertices of the triangle and coinciding with its medians $\overrightarrow{AM}$, $\overrightarrow{BN}$, $\overrightarrow{CP}$.

7. Given the vectors $\vec{a} = \{4, -2, -4\}$, $\vec{b} = \{6, -3, 2\}$. Compute: (1) $\vec{a} \cdot \vec{b}$, (2) $\sqrt{\vec{a}^2}$, (3) $(2\vec{a} - 3\vec{b})(\vec{a} + 2\vec{b})$.

8. Compute the work performed by the force $\vec{f} = \{3, -2, -5\}$ for the time during which the point of its application displaces in rectilinear motion from the position $A\ (2, -3, 5)$ to the position $B\ (3, -2, -1)$.

9. Given the vertices of a quadrilateral $A\ (1, -2, 2)$, $B\ (1, 4, 0)$, $C\ (-4, 1, 1)$, and $D\ (-5, -5, 3)$. Prove that its diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are mutually perpendicular.

10. Determine for what value of α the vectors $\overrightarrow{AB} = \alpha\vec{i} - 3\vec{j} + 2\vec{k}$ and $\overrightarrow{CD} = \vec{i} + 2\vec{j} - \alpha\vec{k}$ are mutually perpendicular.

11. Compute the cosine of the angle φ formed by the vectors $\vec{a} = 2\vec{i} - 4\vec{j} + 4\vec{k}$ and $\vec{b} = -3\vec{i} + 2\vec{j} + 6\vec{k}$.

12. Given the vertices of a triangle: $A\ (-1, -2, 4)$, $B\ (-4, -2, 0)$, and $C\ (3, -2, 1)$. Find the magnitude of its interior angle at the vertex C.

13. Given the vertices of a triangle: $A\ (3, 2, -3)$, $B\ (-5, 1, -1)$, and $C\ (1, -2, 1)$. Find its exterior angle at the vertex C.

14. Prove that the triangle ABC, whose vertices are $A\ (1,\ 2,\ 1)$, $B\ (3,\ -1,\ 7)$, and $C\ (7,\ 4,\ -2)$, is an isosceles triangle, and compute its angles.

15. Write the equation of the plane passing through the point $M_1\ (2,\ 1,\ -1)$ and having the normal vector $\vec{n} = \{1,\ -2,\ 3\}$.

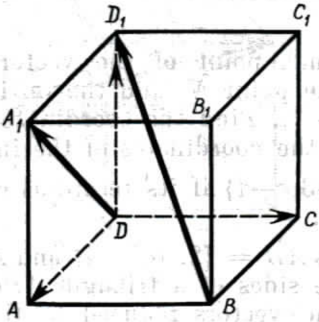


Fig. 130

16. Point $P\ (2,\ -1,\ -1)$ is the foot of the perpendicular dropped from the origin onto a plane. Derive the equation of this plane.

17. Given two points $M_1\ (3,\ -1,\ 2)$ and $M_2\ (4,\ -2,\ -1)$ Write the equation of the plane passing through the point M_1 perpendicular to the vector $\overrightarrow{M_1M_2}$.

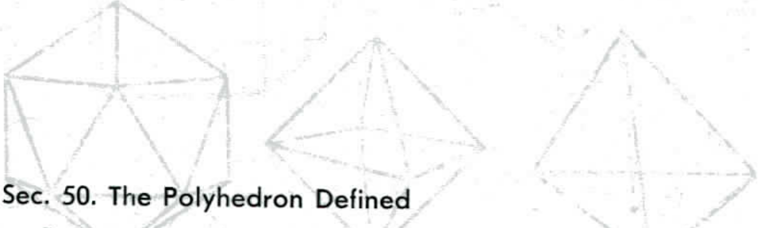
18. Write the equation of the plane which passes through the point $M\ (2,\ -1,\ 1)$ parallel to the plane $2x - y - 1 = 0$.

19. Derive the equation of the plane which passes through the point $M\ (2,\ -3,\ 3)$ parallel to the plane xOy.

20. Find the angle between the diagonal $[BD_1]$ of a cube and the diagonal $[DA_1]$ of its face (Fig. 130).

CHAPTER 7

POLYHEDRA, CYLINDERS, CONES



Sec. 50. The Polyhedron Defined

In our course we shall consider the general properties of polyhedra, and study the properties of prisms and pyramids in more detail.

Definition. A space figure bounded by a surface made up of portions of planes, i.e. of polygons is called the *polyhedron*. Each polygon forming the surface of a polyhedron is called its *face*.

Common edges of two adjacent faces are called the *edges* of a polyhedron, and the vertices of polyhedral angles formed by its faces and meeting at one point are termed the *vertices* of a polyhedron.

A segment of a straight line joining two vertices of a polyhedron not lying on one face is termed the *diagonal*.

We shall consider here only convex polyhedra. It is easy to see that all the faces of a convex polyhedron lie on one side of any plane that contains one of its faces.

The plane containing any face of a polyhedron divides space into two half-spaces. A convex polyhedron belongs only to one of these half-spaces.

Sec. 51. Regular Polyhedra

A *regular* polyhedron is a polyhedron whose faces are congruent regular polygons and whose polyhedral angles are congruent.

At least three faces meet at any vertex of a polyhedron, the sum of their face angles being equal to less than 360°. If the faces of a regular polyhedron are regular triangles, then at any vertex of a polyhedron there can meet either three triangles, $(60^\circ \cdot 3 < 360^\circ)$, or four triangles $(60^\circ \cdot 4 < < 360^\circ)$, or five triangles $(60^\circ \cdot 5 < 360^\circ)$. Six regular triangles do not form a polyhedral angle, since $60^\circ \cdot 6 = 360^\circ$.

Thus, there are only three types of regular polyhedra, whose faces are regular triangles. These are:

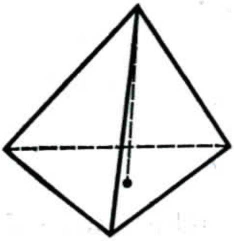


Fig. 131

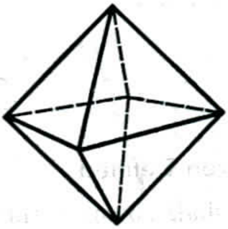


Fig. 132

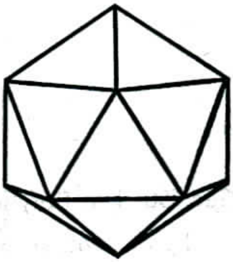


Fig. 133

(1) the *regular tetrahedron* whose surface is made up of four regular triangles (Fig. 131). The tetrahedron has four vertices and six edges;

(2) the *regular octahedron* whose surface consists of eight regular triangles (Fig. 132). The octahedron has six vertices and twelve edges;

(3) the *regular icosahedron* whose surface consists of twenty regular triangles, five triangles meeting at each vertex (Fig. 133). The icosahedron has twelve vertices and thirty edges.

Let us now consider the polyhedron made up of regular quadrilaterals, i.e. of squares. Since $90^\circ \cdot 4 = 360^\circ$, four squares do not form a polyhedral angle. Consequently, there exists only one type of regular polyhedron whose faces are squares which is termed the *cube* (or *hexahedron*). The hexahedron has six faces, eight vertices, and twelve edges (Fig. 134).

There is only one regular polyhedron made up of regular pentagons, in which three faces meet at any of its vertices $(108^\circ \cdot 3 < 360^\circ,\ 108^\circ \cdot 4 > 360^\circ)$. This polyhedron is named the *dodecahedron*. It has twelve faces, twenty vertices, and thirty edges (Fig. 135).

Regular hexagons cannot form even a trihedral angle, since $120^\circ.3 = 360^\circ$.

Thus, we have shown that only regular triangles, squares, and regular pentagons can serve as faces in regular polyhedra, and there are only five regular polyhedra: the tetrahedron, octahedron, icosahedron, hexahedron (or cube), and dodecahedron. The existence of these polyhedra is proved by their construction.

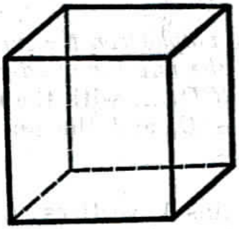


Fig. 134

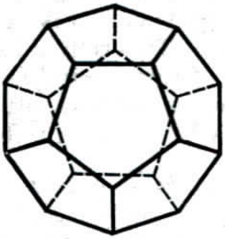


Fig. 135

In the next section we shall get the same result as a corollary of Euler's theorem on polyhedra.

Sec. 52. Euler's Theorem

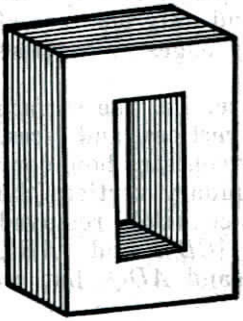


Fig. 136

Let us denote the number of vertices by V, the number of edges by E, and the number of faces by F. Then the above considered regular polyhedra will be characterized by the following numbers: the tetrahedron: $V = 4$, $E = 6$, $F = 4$; the octahedron: $V = 6$, $E = 12$, $F = 8$; the icosahedron: $V = 12$, $E = 30$, $F = 20$; the hexahedron (or cube): $V = 8$, $E = 12$, $F = 6$; the dodecahedron: $V = 20$, $E = 30$, $F = 12$.

Comparing these numbers, we see that for all regular polyhedra $V - E + F = 2$, i.e. the sum of the numbers of vertices and faces exceeds the number of edges by 2.

This formula which is the essence of Euler's theorem turns out to be true for all convex polyhedra and even for a wider class of polyhedra, so-called simple polyhedra.

A *simple polyhedron* is a polyhedron which can be transformed to a sphere with the aid of continuous deformation. The polyhedron shown in Fig. 136 is not a simple polyhedron.

Since in our course we consider only convex polyhedra, let us formulate Euler's theorem particularly for such polyhedra, though its proposition and proof are retained for simple polyhedra as well.

Euler's Theorem. *In any convex polyhedron the sum of the number of vertices and faces exceeds the number of edges by 2.*

Given: a convex polyhedron $ABCDE\ldots$ with the number of vertices V, the number of edges E, and the number of faces F.

Prove: $V - E + F = 2$.

Proof. A closed surface S contains V vertices, E edges, and F faces. Denoting $N = V - E + F$, let us prove that $N = 2$.

In the course of the proof we shall delete successively the faces, one by one, from the surface S.

Let us first remove one of its faces. In the non-closed surface thus obtained (S_1) the number of edges E_1 and the number of vertices V_1 remain unchanged, while the number of faces decreases by one. Consequently, $V_1 - E_1 + F_1 = N - 1$.

We now consider two cases: (1) all faces of the surface S_1 are triangles; (2) among the faces of the surface S_1 there is a polygon with the number of sides $n > 3$.

First case. An edge or a vertex of a polyhedral surface is named *boundary* if it belongs to one and only one face of the surface. A face containing boundary edges or vertices is also named boundary.

Obviously, a closed polyhedral surface, i.e. the surface of a polyhedron has no boundary edges, vertices, and faces. A non-closed (or open) surface necessarily contains boundary edges and faces, but it can have no boundary vertices (the surface illustrated in Fig. 137 with the face $ABCD$ removed has four boundary edges $[AB]$, $[BC]$, $[CD]$, and $[AD]$, four boundary faces ABE, BFC, CDE, and ADQ, but no boundary vertex).

Since each face of a polyhedral surface must have at least one non-boundary edge, a triangular boundary face may have either one, or two boundary edges and at most one boundary vertex.

It is obvious, that a triangular boundary face has a boundary vertex if and only if it has two boundary edges. In this case the boundary vertex is the point of intersection of two boundary edges (in Fig. 138 $A = [AB] \cap [AD]$).

Thus, let us remove from the polyhedral surface S_1 made up from triangular faces the interior of a boundary face together with its boundary elements so that the surface thus obtained remains polyhedral. This surface (S_2) has V_2 vertices, E_2 edges, and F_2 faces. Let us compute the quantity $V_2 - E_2 + F_2$.

If the removed face had only one boundary edge, then it had no boundary vertices, and $V_2 - E_2 + F_2 = V_1 - (E_1 - 1) + (F_1 - 1) = V_1 - E_1 + F_1 = N - 1$. But if

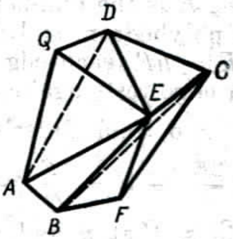


Fig. 137

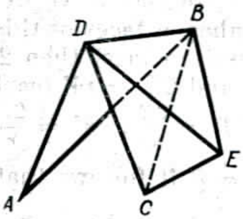


Fig. 138

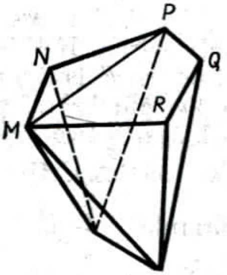


Fig. 139

the removed face had two boundary edges, then it had one boundary vertex, and $V_2 - E_2 + F_2 = V_1 - 1 - (E_1 - 2) + (F_1 - 1) = V_1 - E_1 + F_1 = N - 1$.

Thus, the removal of the interior of one boundary face together with its boundary vertices and edges does not change the magnitude of the expression $V_1 - E_1 + F_1$.

Let us now delete from the surface S_2 the interior of one of its boundary faces together with its boundary edges and vertices. For the remaining polyhedral surface S_3 we get $V_3 - E_3 + F_3 = N - 1$.

We shall continue removing the boundary faces one by one until only one face is left—a triangle in which $V - E + F = 1$. Since with each removal the quantity $V_k - E_k + F_k$ has remained unchanged and equal to $N - 1$, we have $N - 1 = 1$, i.e. $N = 2$, which was required to be proved.

Second case. The surface S_1 has a face with the number of sides $n > 3$. Let us draw in this face a diagonal which is considered to be an "edge". This "edge" divides one face into two: *MNP* and *MPQR* (Fig. 139). The "surface" thus obtained is characterized by the number of faces and by the number of "edges" increased by one, the number of vertices remaining unchanged. Consequently, the quantity $V_1 - E_1 + F_1$ remains unchanged.

Thus, dividing each non-triangular face into triangular ones (for the proof it is not significant that some "faces" lie in one plane), we come to the first case without changing the magnitude of the expression $V_1 - E_1 + F_1$. And so, the theorem has been proved completely.

Corollary. *There exist not more than five types of regular polyhedra.*

Proof. Suppose r edges meet at each vertex of a regular polyhedron. If V is the number of vertices, E is the number of edges, F is the number of faces of this polyhedron, and n is the number of sides in a face, then $2E = nF$ (each edge belongs to two faces) and $2E = rV$ (each edge rests against two vertices). Hence $F = \frac{2E}{n}$, $V = \frac{2E}{r}$ and from Euler's formula $V - E + F = 2$ it follows that $\frac{2E}{r} - E + \frac{2E}{n} = 2$, or $\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E}$. It is obvious, that n and r cannot simultaneously exceed 3, since in that case $\frac{1}{n} + \frac{1}{r} \leqslant \frac{1}{2}$, which is impossible for $E > 0$. Thus, we have only to find out what value is attained by r for $n = 3$, and by n for $r = 3$.

If $n = 3$, then $\frac{1}{r} - \frac{1}{6} = \frac{1}{E}$, whence it is seen that r can attain only the values 3, 4, 5. Then E is equal to 6, 12, and 30, respectively.

If $r = 3$, then $\frac{1}{n} - \frac{1}{6} = \frac{1}{E}$ and n takes on only the values 3, 4, and 5, E being equal to 6, 12, and 30, respectively.

Thus, there are only five regular polyherda: the tetrahedron ($n = 3$, $r = 3$), the octahedron ($n = 3$, $r = 4$), the icosahedron; ($n = 3$, $r = 5$), the cube ($n = 4$, $r = 3$), and dodecahedron ($n = 5$, $r = 3$).

Sec. 53. The Prism

Let there be given in a plane α a polygonal line $ABCD \ldots Q$ and a straight line l intersecting the plane α (Fig. 140).

Definition 1. A surface generated by a moving straight line MN which continuously displaces along a polygonal line $ABCD \dots Q$ parallel to a straight line l is termed *prismatic*. The moving line is called the *generatrix*, and the polygonal line $ABCD \dots Q$ is the *directrix* (Fig. 140).

A prismatic surface is said to be open (or non-closed) or closed depending on whether its directrix is open or closed.

A prismatic surface consists of plane "stripes" intersecting along parallel straight lines which are called elements (AA_1), (BB_1), (CC_1), . . . (Fig. 140).

Definition 2. The *prism* is a space figure bounded by a closed prismatic surface and intersected by two parallel planes. The prismatic surface bounding the prism is called the *lateral surface* of the prism, and the polygons lying in the parallel planes are termed the *bases* of the prism (Fig. 141).

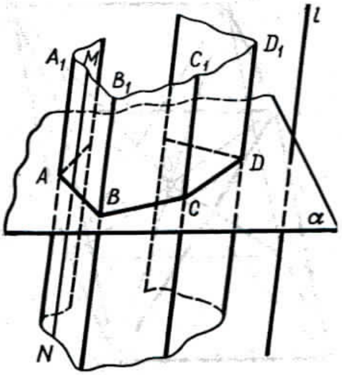


Fig. 140

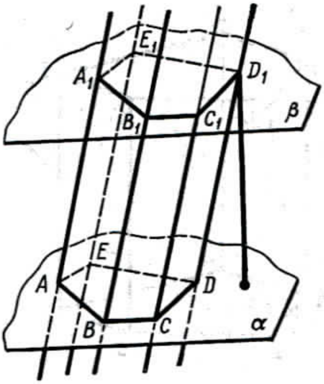


Fig. 141

The *altitude* of a prism is defined as the perpendicular distance between the bases. Or in other words, the altitude is a perpendicular dropped from any point of one base onto the plane containing the other base (Fig. 141).

From the definition of the prism it follows that its lateral surface consists of quadrilaterals (lateral faces) obtained as a result of intersection of the plane "stripes" forming the prismatic surface by two parallel planes.

Theorem 1. *The lateral faces of a prism are parallelograms; the bases of a prism are congruent polygons with respectively parallel sides.*

Given: a prism $ABCDEA_1B_1C_1D_1E_1$ (Fig. 141).

Prove: ABB_1A_1, BCC_1B_1, . . . are parallelograms, $ABCDE \cong A_1B_1C_1D_1E_1$, $[AB] \parallel [A_1B_1]$, $[BC] \parallel [B_1C_1]$, ...

Proof. Let us consider, for instance, the lateral face ABA_1B_1. The lateral edges $[AA_1]$ and $[BB_1]$ are parallel to each other as two elements of a prismatic surface, and $[AA_1]$ is congruent to $[BB_1]$ as segments of parallel lines contained between two parallel planes. Consequently, ABB_1A_1 is a parallelogram, and $[AB]$ is congruent and parallel to $[A_1B_1]$. The congruence and parallelism of the edges $[BC]$ and $[B_1C_1]$, $[CD]$ and $[C_1D_1]$, $[DE]$ and $[D_1E_1]$ are proved in a similar way. Besides, the angle ABC is congruent to the angle $A_1B_1C_1$, $\angle BCD$ is congruent to $\angle B_1C_1D_1$, . . ., as angles with parallel sides of the same direction. Therefore, $ABCDE$ is congruent to $A_1B_1C_1D_1E_1$, which was required to be proved.

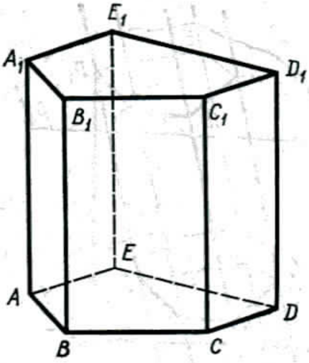


Fig. 142

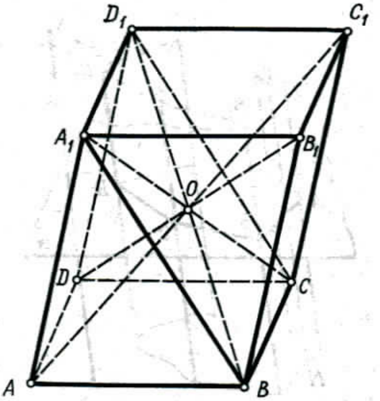


Fig. 143

A prism is called a *right* one if its lateral edges are perpendicular to the planes containing the bases. Obviously, in this case each lateral edge is the altitude of the prism, the lateral faces being rectangles (Fig. 142).

A right prism whose bases are regular polygons is called *regular* (a regular prism, excepting the cube, is not a regular polyhedron; see Sec. 51).

A prism is named triangular, quadrangular, etc. depending on what polygon serves as its base. A prism with a triangle as base is a triangular prism; one with a quadrilateral as base is a quadrangular prism; etc. A prism whose bases are parallelograms is called the *parallelepiped.* The parallelepiped whose lateral edges are perpendicular to the planes containing its bases is termed a *right* parallelepiped. It is obvious that all of its lateral faces are rectangles.

A *rectangular* parallelepiped is defined as a right parallelepiped whose bases are rectangles.

The lengths of the edges of a rectangular parallelepiped meeting at one point are called its *dimensions.*

A rectangular parallelepiped with all three dimensions equal is a *cube.*

Theorem 2. *In any parallelepiped:* (1) *the opposite faces are congruent and parallel to each other*; (2) *the diagonals intersect at one point and are bisected by this point.*

Given: a parallelepiped $ABCDA_1B_1C_1D_1$ and its diagonals $[AC_1]$, $[BD_1]$, $[DB_1]$, and $[CA_1]$.

Prove: AA_1BB_1 is congruent and parallel to DCC_1D_1, ADD_1A_1 is congruent and parallel to BCC_1B_1; $[AC_1] \cap [BD_1] \cap [DB_1] \cap [CA_1] = 0$ and $|AO| = |OC_1|$, $|DO| = |OB_1|$, . . . (Fig. 143).

Proof. (1) Since $|AD| = |BC|$, $|AA_1| = |BB_1|$ and $[AD]$ is parallel to $[BC]$ and $[BB_1]$ to $[AA_1]$, the parallelograms ADD_1A_1 and BCC_1B_1 are congruent and parallel. By the same reasons ABB_1A_1 is congruent and parallel to DCC_1D_1.

(2) We join A_1 to B and D_1 to C. The figure thus obtained is a parallelogram, since the sides $[A_1D_1]$ and $[BC]$ are equal and parallel. The diagonals $[A_1C]$ and $[BD_1]$ of the parallelepiped are diagonals of the parallelogram obtained, consequently, they are bisected in the point of intersection. Proceeding in the same way, we take the diagonal $[BD_1]$ and a third diagonal, say $[AC_1]$, and show that at the point of intersection they are bisected. Hence, the diagonal $[AC_1]$ passes through the mid-point of the diagonal $[BD_1]$, i.e. through the point O. Analogously, considering the parallelogram ADC_1B_1 we see that $[DB_1]$ also passes through the point O and is bisected by this point.

Theorem 3. *The squared length of the diagonal of a rectangular parallelepiped is equal to the sum of squares of its three dimensions.*

Given: a rectangular parallelepiped $ABCDA_1B_1C_1D_1$, $|AD| = a$, $|DC| = b$, $|DD_1| = c$.

Prove: $|BD_1| = a^2 + b^2 + c^2$ (Fig. 144)

Proof. Let us take D for the origin and lay off the vector $\vec{i}$ on (AD), vector $\vec{j}$ on (DC), and vector $\vec{k}$ on (DD_1). Then

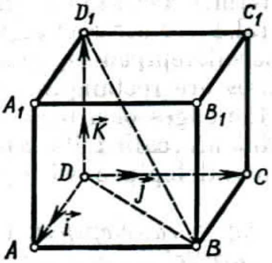


Fig. 144

$\overrightarrow{DA} = a\vec{i}$, $\overrightarrow{DC} = b\vec{j}$, and $\overrightarrow{DD_1} = c\vec{k}$. Obviously, $\overrightarrow{BD_1} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DD_1} = -\overrightarrow{DA} - \overrightarrow{DC} + \overrightarrow{DD_1} = -a\vec{i} - b\vec{j} + c\vec{k} = \{-a, -b, c\}$.

Hence,

$$|\overrightarrow{BD_1}|^2 = a^2 + b^2 + c^2.$$

Corollary. *All the diagonals of a rectangular parallelepiped are congruent.*

Sec. 54. A Cylindrical Surface. The Cylinder

A right circular cylinder is a space figure bounded by two congruent and parallel circles (bases) and a lateral surface which is called cylindrical.

A right circular cylinder can be generated by rotating a rectangle about one of its sides. During this rotation the opposite side generates the lateral surface of the cylinder.

The rotation of this side can be considered as its continuous displacement along the directrix (the circle L) parallel to the straight line (BC) (Fig. 145).

Comparing a right prism with a right circular cylinder, we see that they possess the following properties in common:

(1) the bases of these figures are congruent and lie in parallel planes: polygons in a prism, and circles in a right circular cylinder;

(2) the lateral surfaces of these figures are obtained as a result of a continuous displacement of a straight line (the generatrix) along a plane curve: a polygonal line or a circle.

Let us now consider the general case.

Suppose MN is an arbitrary line, closed or open, lying in a plane α, and l is a straight line piercing this plane (Fig. 146).

Definition 1. The surface generated by a moving straight line (AB), which continuously displaces along the line MN and parallel to the axis l is called *cylindrical*. The moving line is termed the *generatrix*, and the line MN is the *directrix* of this surface.

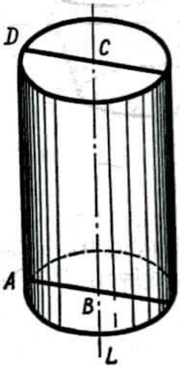


Fig. 145

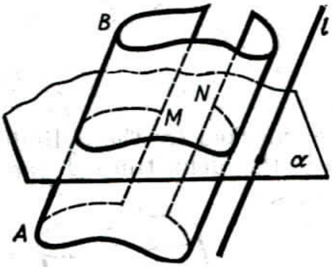


Fig. 146

A cylindrical surface is said to be closed or non-closed (open) depending on whether its directrix is closed or open.

Let us consider a few examples of cylindrical surfaces.

1. The directrix is a straight line (Fig. 147). In this case the cylindrical surface is a plane passing through the directrix parallel to the axis l.

2. The directrix is a polygonal line (see Fig. 140). Here we get a prismatic surface. Consequently, a prismatic surface is a particular case of the cylindrical surface.

3. The directrix is a circle. The corresponding cylindrical surface is called circular (Fig. 148).

Definition 2. The *cylindrical solid* (or *cylinder*) is a space figure bounded by a closed cylindrical surface cut by two parallel planes (Fig. 149).

The cylindrical surface bounding a cylinder is called the *lateral surface* of the cylinder, and the portions of the parallel planes marked by the cylindrical surface its *bases* (compare with the definition of the prism given in Sec. 53).

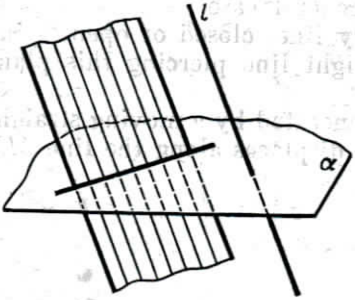


Fig. 147

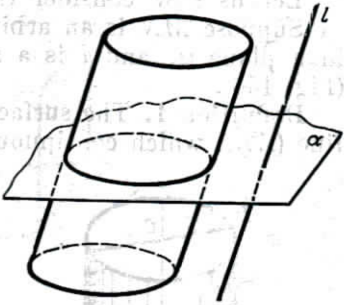


Fig. 148

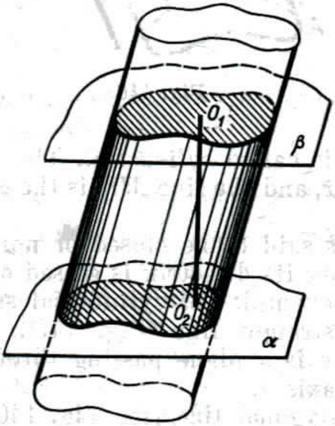


Fig. 149

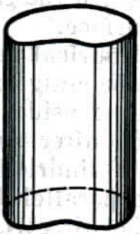


Fig. 150

The *altitude* of a cylinder is defined as a perpendicular $[O_1O_2]$ dropped from any point of one base onto the plane containing the other base (Fig. 149).

A cylinder is called a *right* cylinder if its elements are perpendicular to the planes containing the bases (Fig. 150), otherwise it is an *oblique* cylinder.

It is obvious, that each element of a right cylinder is its altitude.

Cylindrical solids differ in the kind of their bases.

1. A right cylinder whose bases are circles is called a *right circular cylinder*. The latter is sometimes called simply a cylinder (to avoid possible "confusion" we have introduced the notion of a cylindrical solid meaning a cylinder of an arbitrary form).
2. A right cylinder whose bases are ellipses is called a *right elliptic cylinder* (or simply an *elliptic* cylinder).
3. A right cylinder whose bases are polygons is a *right prism*.
4. An oblique cylinder whose bases are polygons is called an *oblique prism*, and so on.

Note. Unification of such, one would think different, solids as the prism and the circular cylinder in one class of cylindrical solids might seem artificial. But later on (in Chapter 9) we shall see the advantage of our classification.

Sec. 55. The Pyramid

A *pyramid* is a polyhedron with one face a polygon and the other faces triangles with a common vertex. The polygon is the base of the pyramid and the triangles are the lateral faces (see Fig. 151).

The pyramid may also be considered as the intersection of a polyhedral angle with the half-space containing the vertex of this angle, provided the plane bounding this half-space cuts all the edges of the polyhedral angle (Fig. 151).

As is known, the *altitude* of a pyramid is a segment of the perpendicular dropped from the vertex of the pyramid onto the plane containing its base.

A pyramid is called a *right* pyramid if its base is a regular polygon and the altitude passes through the centre of the base. Obviously, the lateral faces of a regular pyramid are isosceles and congruent triangles. The altitudes of these triangles (which are also congruent) are called the *slant heights* of a regular pyramid (Fig. 152).

Like prisms, a pyramid with triangle as base is called a triangular pyramid; one with a quadrilateral as base is called a quadrangular pyramid; etc.

In the next section we shall establish connection between a pyramid and a cone. For this purpose let us introduce one more definition of the pyramid.

Definition 1. Let $ABCD \ldots H$ be a closed polygonal line lying in a plane α, and S is a point not belonging to α. The surface generated by a moving ray $[SA)$ which in the course of its continuous displacement intersects the polygonal line $ABCD \ldots H$ (i.e. the directrix) is called a *closed pyramidal surface* (Fig. 151).

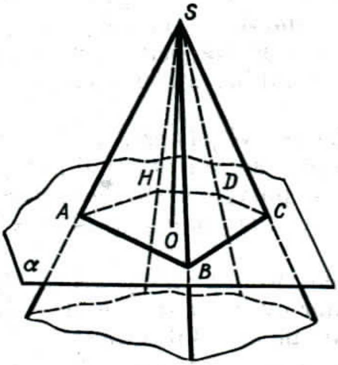


Fig. 151

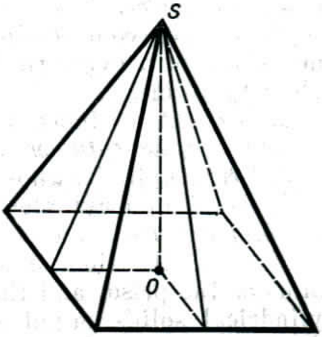


Fig. 152

Definition 2. A *pyramid* is a space figure bounded by a closed pyramidal surface and a plane which cuts all of its elements (Fig. 151).

Sec. 56. A Conical Surface. The Cone

A *cone* can be obtained by rotating a right-angled triangle about one of its legs (Fig. 153). During this rotation the hypotenuse generates the lateral surface of the cone. The rotation of the hypotenuse may be considered as its continuous motion such that in its course it intersects the circle AD and has one fixed point (B). Comparing a pyramid with a circular cone, we see that they possess a number of properties in common:

(1) both figures have plane bases;

(2) the lateral surfaces of these figures are generated by a continuous displacement of a straight line passing through the vertex and intersecting the boundary of the base.

Bearing in mind the same reasoning as above (see Note in Sec. 54), let us introduce the notions of an arbitrary conical surface and an arbitrary cone.

Definition 1. Let a line MN lie in a plane α, and a point S be situated outside the plane α. The surface generated by a moving straight line which continuously displaces passing through the point S and intersecting a plane curve MN is called *conical* (Fig. 154). The moving straight line is termed the *generatrix*, the point S is the *vertex*, and the curve MN is the *directrix* of the conical surfaces.

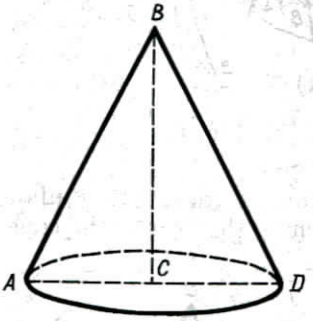


Fig. 153

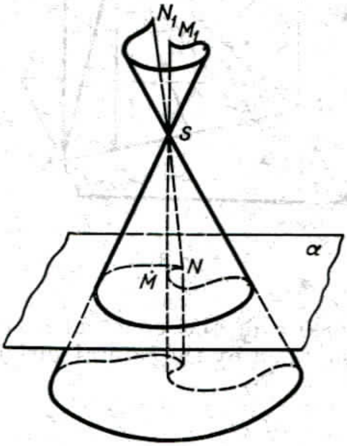


Fig. 154

The directrix may be closed and non-closed (or open). Accordingly, the conical surface is called closed or non-closed (open).

Let us consider a few examples of conical surfaces.

1. The directrix is a straight line (MN). In this case the generatrix displaces in the plane β, which passes through (MN) and the point S (Fig. 155).

2. The directrix is a polygonal line, closed (Fig. 156, *a*) or non-closed (Fig. 156, *b*). The conical surface obtained in this case is called pyramidal (see Sec. 55).

3. The directrix is a circle. In this case the conical surface is said to be circular (Fig. 157).

Definition 2. A *cone* is a space figure bounded by a portion of a closed conical surface located on one side of the vertex S and a plane intersecting all of its elements (Fig. 158). The conical surface bounding the cone is called the *lateral*

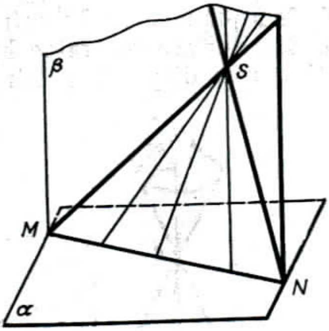


Fig. 155

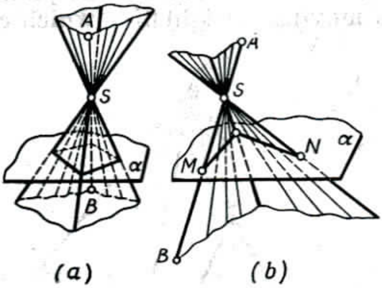


Fig. 156

surface of the cone, and the portion of the cutting plane bounded by the lateral surface is termed the *base* of the cone.

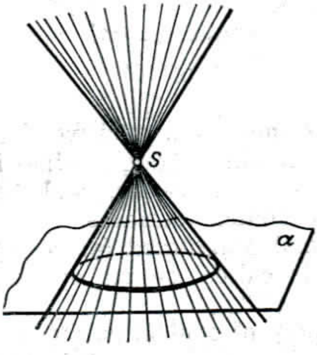


Fig. 157

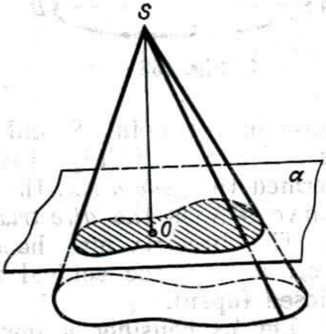


Fig. 158

A segment of the perpendicular $[SO]$ dropped from the vertex of the cone onto the plane containing the base is called the *altitude* of the cone (Fig. 158).

Cones differ in their bases. A cone whose base is a polygon is called the *pyramid*. A cone whose base is a circle and the altitude passes through the centre of the circle is called a *circular* cone.

Sec. 57. Homothety in Space

Let us introduce the notion of homothety in space.

Definition. *Homothety in space* with the centre of similitude O and the ratio of similitude $k \neq 0$ is an into transformation of space such that any point M is transformed into the point M_1, i.e. $M_1 = H_O^k(M)$, so that $\overrightarrow{OM_1} = k\cdot\overrightarrow{OM}$.

Theorem 1. *In homothety a plane is transformed into a parallel plane.*

Given: a plane α, and its homothetic transformation $H_O^k(\alpha)$.

Prove: $H_O^k(\alpha)$ is a plane parallel to α.

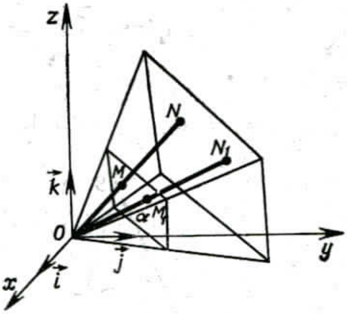


Fig. 159

Proof. Let us select a rectangular coordinate system in space so that its origin coincides with the centre of similitude (Fig. 159). We then write the equation of the plane α in this coordinate system:

$$Ax + By + Cz + D = 0. \qquad (1)$$

Let $M(x, y, z)$ be an arbitrarily chosen point in the plane α and $N(x_1, y_1, z_1)$ its homothety. Since $\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$, $\overrightarrow{ON} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ and $\overrightarrow{ON} = k\cdot\overrightarrow{OM}$, we have $x_1 = kx$, $y_1 = ky$, $z_1 = kz$. Obtaining the expressions for x, y, and z and substituting them in the equation (1), we get the following equation:

$$Ax_1 + By_1 + Cz_1 + kD = 0, \qquad (2)$$

which is satisfied by the coordinates of the image of any point lying in plane α. Obviously, (2) is the equation of a plane β which is parallel to the plane α. Thus, a homothetic transformation of any point $M \in \alpha$ lies in the plane β which is parallel to α.

Let us show that any point in the plane β is the image of some point belonging to the plane α. Indeed, let $N_1(x_2, y_2, z_2) \in \beta$, i.e.

$$Ax_2 + By_2 + Cz_2 + kD = 0.$$

Dividing all terms of this equation by k, we get $A\,\frac{x_2}{k} + B\,\frac{y_2}{k} + C\,\frac{z_2}{k} + D = 0$, whence it follows that the point $M_1\left(\frac{x_2}{k}, \frac{y_2}{k}, \frac{z_2}{k}\right)$ satisfies the equation (1), i.e. it lies in the plane α. Since $\overrightarrow{ON_1} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$, $\overrightarrow{OM}_1 = \frac{x_2}{k}\vec{i} + \frac{y_2}{k}\vec{j} + \frac{z_2}{k}\vec{k}$, then $\overrightarrow{ON_1} = k\cdot\overrightarrow{OM_1}$, i.e. $N_1 = H_O^k(M_1)$, which was required to be proved. Homothety in space is characterized by the following properties.

1°. *The centre of similitude is mapped into itself.*

2°. $|A_1B_1| = k\cdot|AB|$, where $A_1 = H_O^k(A)$, $B_1 = H_O^k(B)$.

3°. *Any ray is transformed into a ray of the same direction if* $k > 0$, *and into an oppositely directed ray if* $k < 0$.

4°. *The ratio of the areas of homothetic polygons is equal to the square of the ratio of similitude.*

The proofs of all of these properties we leave to the reader.

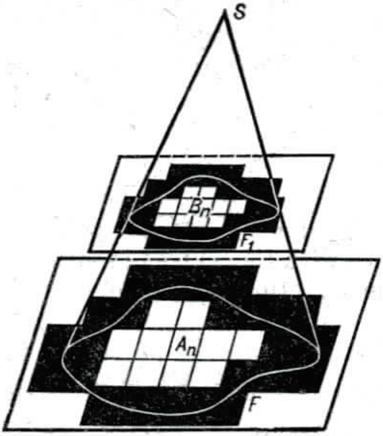


Fig. 160

Theorem 2. *The ratio of the areas of plane homothetic figures is equal to the square of the ratio of similitude.*

Given: a plane figure F and its homothetic transformation $F_1 = H\frac{k}{S}(F)$.

Prove: area of $F_1 = k^2\cdot$area of F.

Proof. For polygons this property is formulated in 4°. Let us prove it for arbitrary figures.

According to Theorem 1, the figures F and F_1 lie in parallel planes, as is shown in Fig. 160.

We take a measuring grid consisting of squares with the side of length $1/10^n$ and lay it on the figure F to single out, as in Sec. 42, the figures A_n and B_n made up of the squares of the measuring grid: $A_n \subset F \subset B_n$ (Fig. 160).

Homothety with the centre of similitude S and the ratio of similitude $k = \frac{|SD|}{|SC|} > 0$ transforms a square into a square (properties 2° and 3°); therefore the images $H_S^k(A_n)$ and $H_S^k(B_n)$ will be made up of squares with the side of length $k/10^n$ (property 2°). Obviously, area $H_S^k(A_n) = k^2 \cdot$ area A_n, area $H_S^k(B_n) = k^2 \cdot$ area B_n and since $H_S^k(A_n) \subset H_S^k(F) \subset$ $\subset H_S^k(B_n)$, we have

$$k^2 \cdot \text{area } A_n < \text{area } H_S^k(F) < k^2 \cdot \text{area } B_n. \qquad (3)$$

As n increases infinitely, the quantities 'area A_n' and 'area B_n' approach 'area F' as a limit: $\lim\limits_{n\to\infty}$ area $A_n =$ $= \lim\limits_{n\to\infty}$ area B_n = area F. Consequently, the extremes of the double inequality (3) tend to one and the same limit, therefore area $H_S^k(F) = k^2 \cdot$ area F, which was required to be proved.

Sec. 58. Properties of Parallel Sections of a Cone (Pyramid). Frustums of a Cone (Pyramid)

Theorem. *If a cone* is cut by a plane parallel to its base, then the section thus obtained is a figure homothetic to the base with the centre of similitude at the vertex of the cone.*

Given: a cone T, its base F, a plane α parallel to the plane F containing the base; $F_1 = T \cap \alpha$ (Fig. 161).

Prove: $F_1 = H_S^k(F)$, $k = \frac{|SO_1|}{|SO|}$.

Proof. Let us consider the homothetic transformation with the centre of similitude at the vertex S of the cone and the ratio of similitude $k = \frac{|SO_1|}{|SO|}$, where $O_1 = \alpha \cap [SO]$ (Fig. 161). This homothety transforms the point O into O_1, and the plane containing the lower base into a plane parallel to this plane and passing through the point O_1, i.e. into the plane α. Thus, the homothetic transformation of any point M belonging to F belongs both to the ray $[SM)$ and to the plane α, i.e.

$$M_1 = H_S^k(M) \in [SM) \cap \alpha \in F_1.$$

---

* We mean an arbitrary cone here.

On the other hand, any point $N_1 \in F_1$ is a homothetic transformation of the point $N = F \cap [SN_1)$. Thus, $F_1 = H_S^k(F)$, which was required to be proved.

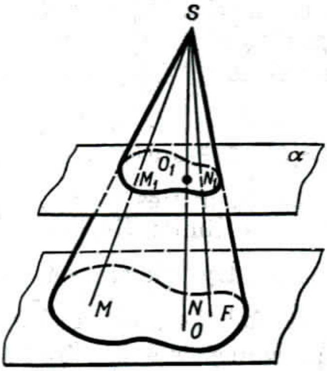


Fig. 161

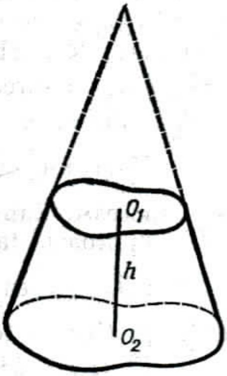


Fig. 162

Corollary. *If a cone is cut by a plane which is parallel to its base, then the areas of the section and the base are to each other as the squares of their distances from the vertex of the cone.*

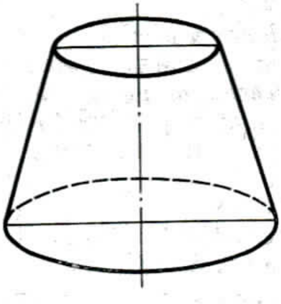


Fig. 163

The validity of this corollary follows from Theorems 1 and 2 of Sec. 57, and the theorem of Sec. 58, the ratio of similitude being equal to $\frac{|SO_1|}{|SO|}$.

Definition. The part of the cone (pyramid) bounded by the base and a cutting plane parallel to the base is called the *frustum of a cone* (*pyramid*). As it follows from the theorem proved above, the bases of the frustum are homothetic (Fig. 162).

The *altitude* of the frustum of a cone is defined as a segment of the perpendicular dropped from any point of one base onto the plane containing the other base (Fig. 162).

A frustum of a pyramid is called *regular* if its bases are regular polygons and the line segment joining the centres of the bases is the altitude of the frustum. As is obvious, a regular frustum is the part of a regular pyramid.

The bases of a circular frustum of a cone are circles, and the line segment joining the centres of these circles is the altitude of this frustum (Fig. 163). A circular frustum of a cone can be obtained by rotating an isosceles trapezium about its axis of symmetry (Fig. 163).

Sec. 59. Sections of Polyhedra

Let a space figure Φ be cut by a plane α. Then their intersection is a plane figure F which is called the *section*: $F = \alpha \cap \Phi$.

In particular, if the space figure is a polyhedron, then the section is a polygon. To construct the section of a polyhedron, we have to find the line segments along which the faces of the polyhedron are intersected by the cutting plane.

Let us consider a few methods of constructing sections.

Example 1. Given a cube $ABCDA_1B_1C_1D_1$. A cutting plane is drawn through the edge $[AB]$ of the base and the centre E of the face CDD_1C_1. Construct the section.

Solution. Since $[AB]$ is parallel to (DCC_1D_1), the cutting plane α intersects DCC_1D_1 along the line segment $[MN]$ passing through the point E parallel to $[AB]$ or $[CD]$ (Figure 164). Joining the points A, $M = \alpha \cap [DD_1]$, $N = \alpha \cap [CC_1]$, and B, we get a rectangle $AMNB$ which is the required section.

Example 2. A cutting plane is drawn through the vertex A of the lower base of the cube $ABCDA_1B_1C_1D_1$ and the centres M and N of two lateral faces not containing A. Construct the section.

Solution. Since $[MN]$ is parallel to $(ABCD)$, the cutting plane α intersects $(ABCD)$ along a straight line $p = \alpha \cap (ABCD)$ which passes through the point A parallel to $[MN]$ (Fig. 165). Let us find the point of intersection of the line p with the plane (BCC_1B_1). To do this we draw (BC) to intersect $p : P = (BC) \cap p$. Since the points M and P belong to α, then $(MP) \subset \alpha$, and hence, both $E = (MP) \cap (BB_1)$ and $F = (MP) \cap (CC_1)$ also belong to α. Now we easily find the intersection of α with the face $CC_1D_1D : H = (DD_1) \cap (FN)$.

The above method of constructing sections consists in that we extend the cutting plane beyond the limits of a polyhedron and find its intersection with the planes containing the lateral faces (but not only with the faces themselves).

If we have to compute the area of a section or to solve some other computational problem, we must locate more exactly the vertices of the section. Thus, in Example 2, solving the triangle PFC (Fig. 165) and putting $|AB| = a$,

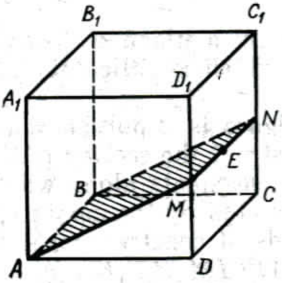


Fig. 164

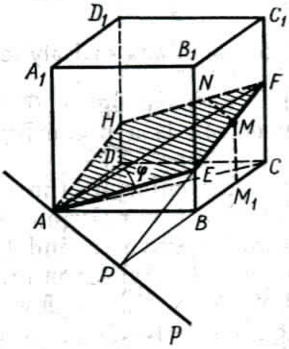


Fig. 165

we find: $|MM_1| = a/2$, $|BM_1| = |M_1C| = a/2$, $|PC| = 2a$ (in $\triangle ABP\ \hat{B} = 90°$, $\hat{A} = 45°$, $|PB| = |AB| = a$). Furthermore, since $|CF| / |M_1M| = |PC| / |PM_1|$, then $|CF| = 2a/3$, where $|BE| = a/3$. Since $ABCD$ is an orthogonal projection of the section $AEFH$, we have $S_{AEFH} = \frac{S_{ABCD}}{\cos\varphi}$, where $\varphi = \widehat{FAC}$. But $\tan\varphi = \frac{|FC}{AC|} = \frac{2a/3}{a\sqrt{2}} = \frac{3}{3\sqrt{2}}$, whence $\cos\varphi = \frac{3}{\sqrt{11}}$. Hence, $S_{AEFH} = \frac{a^2\sqrt{11}}{3}$.

Example 3. A quadrangular prism $ABCDA_1B_1C_1D_1$ is intersected by a plane passing through the points M, N, and P located on the lateral edges of the prism. Construct the section figure (Fig. 166).

Solution. If $|CP| = |BN| = |MA|$, then the cutting plane is parallel to the bases and the section figure is a quadrilateral congruent to the bases.

Consider the general case. Suppose $|CP| < |BN|$ and $|AM| < |BN|$. Let us find the intersection of the secant plane α with $(ABCD)$. For this purpose we extend the lines (BC) and (NP) to get the point F of their intersection and also the lines (NM) and (BA) to get the point E of their intersection: $F = (BC) \cap (NP)$, $E = (BA) \cap (NM)$.

Since $E \in \alpha \cap (ABCD)$ and $F \in \alpha \cap (ABCD)$, we have $(EF) \in \alpha \cap (ABCD)$. If (EF) intersects the face $ABCD$, then $[QH] = (EF) \cap ABCD$ is a side of the required section, and joining the points Q, M, P and H, we get the pentagon $MQHPN$.

If (EF) does not intersect $ABCD$, then the cutting plane α intersects the edge DD_1 (Fig. 167). Let us denote the point of their intersection by H: $H = \alpha \cap [DD_1]$ and find it. The points N and H belong to the diagonal section DBB_1D_1 whose plane will intersect (EF) at point $Q = (BD) \cap (EF)$. Since $Q \in \alpha$ and $N \in \alpha$, then $[NQ] \in \alpha$, consequently, $H = [NQ] \cap [DD_1]$ also belongs to α. Thus, we have found the point of intersection of the cutting plane with the edge $[DD_1]$. Joining the points M, N, P, and H, we get the required section, i.e. the quadrilateral $MNPH$.

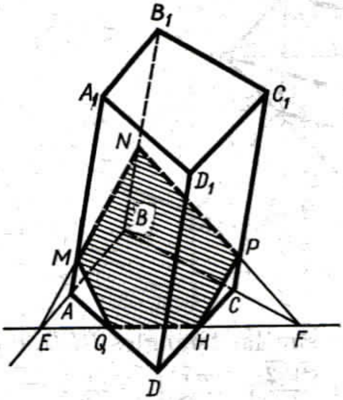


Fig. 166

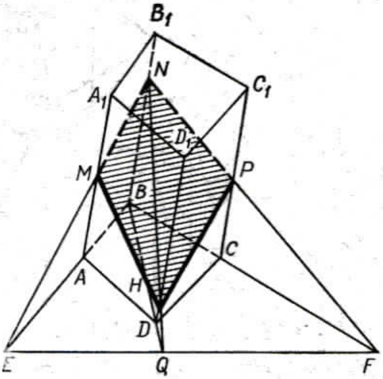


Fig. 167

The coordinate method (see Problems 5 and 6 of Sec. 49) can also be used for finding section figures.

Sec. 60. Worked Problems

Problem 1. A cube is inscribed into an octahedron so that all its vertices are found on the edges of the octahedron (Fig. 168). The edge of the octahedron is equal to a. Determine the edge of the cube.

Solution. The section $ABFD$ of the octahedron is a square, since $|AB| = |BF| = |FD] = |DA| = a$ and $|BD| = |AF|$. In this section $[MN]$ is the diagonal of the base of the cube. Denoting the edge of the cube by x, we have:

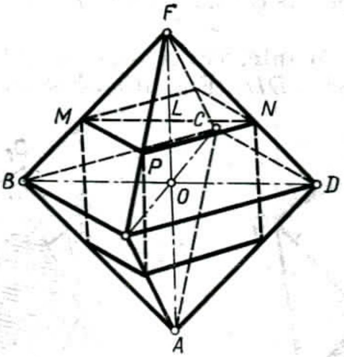


Fig. 168

$|MN| = x\sqrt{2}$, $|MP| = x$. In similar triangles BFD and MFN: $|BD| = a\sqrt{2}$, $|FO| = \frac{a\sqrt{2}}{2}$, $|FL| = \frac{a\sqrt{2}}{2} - \frac{x}{2}$. Since $\frac{|BD|}{|MN|} = \frac{|OF|}{|FL|}$, we have $\frac{a\sqrt{2}}{x\sqrt{2}} = \frac{a\sqrt{2}}{a\sqrt{2}-x}$, or $a\sqrt{2} - x = x\sqrt{2}$, whence

$$x = \frac{a\sqrt{2}}{\sqrt{2}+1} = a(2-\sqrt{2}).$$

Problem 2. In a regular triangular prism with the side of the base equal to a and the altitude equal to H, a plane is drawn through the edge of the lower base and at an angle φ to the latter. Find the area of the section.

Solution. The cutting plane passes through the edge $[AB]$ and, depending on the magnitude of the angle φ, intersects (Fig. 169) or does not intersect (Fig. 170) the lateral edge $[CC_1]$.

1. If $\tan\varphi \leqslant \frac{|CC_1|}{|DC|} = \frac{2H}{a\sqrt{3}}$, then the cutting plane intersects the edge $[CC_1]$ at a point E, and, consequently, the face ACC_1A_1 is cut along the line segment $[AE]$ and

the face BCC_1B_1 along the segment $[BE]$. In this case the required section is a triangle ABE. Since the triangle ABC is an orthogonal projection of the triangle ABE, we have

$$\text{area } ABE = \frac{\text{area } ABC}{\cos\varphi} = \frac{a^2\sqrt{3}}{4\cos\varphi}.$$

2. If $\tan\varphi > \frac{2H}{a\sqrt{3}}$, then the secant plane cuts the upper base $A_1B_1C_1$ along a straight line $[MN]$ parallel to $[AB]$ (Fig. 170). The section figure is a trapezium $ABNM$. Since the orthogonal projection of $ABNM$ on the plane containing the lower base is the trapezium $ABPQ$, we have

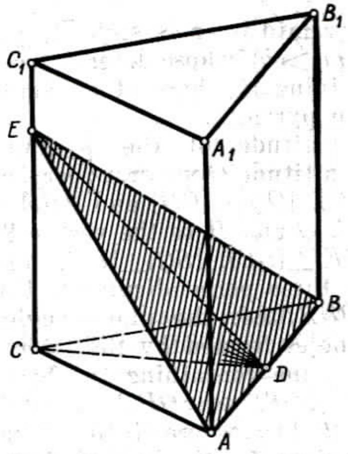


Fig. 169

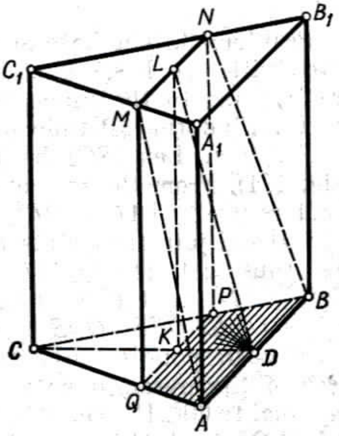


Fig. 170

$$\text{area } ABNM = \frac{\text{area } ABPQ}{\cos\varphi}. \tag{1}$$

In notation of Fig. 170 we have: area $ABPQ = \frac{|AB| + |QP|}{2} \times$ $\times |KD|$.

The unknown quantities $|KD|$ and $|QP|$ are found from the triangles DKL and ABC. In the triangle DKL we know the leg $|KL| = H$ and the angle φ. Hence, $|DK| = H\cot\varphi$. In the triangle ABC $[QP]$ is parallel to $[AB]$. Hence, $|AB| \,\|\, |QP| = |CD| \,\|\, |KC|$, whence

$$|QP| = \frac{|AB|\cdot|KC|}{|CD|} = \frac{a\left(\frac{a\sqrt{3}}{2} - H\cot\varphi\right)}{a\sqrt{3}/2} = \frac{3a - 2\sqrt{3}\,H\cot\varphi}{3}.$$

Returning to the equation (1), we find

$$\text{area } ABNM = \frac{1}{\cos\varphi}\cdot\frac{1}{2}\left[a + \frac{3a - 2\sqrt{3}\,H\cot\varphi}{3}\right]H\cot\varphi = \\ = \frac{(3a - \sqrt{3}\,H\cot\varphi)\,H}{3\sin\varphi}.$$

Problem 3. The base of a pyramid is an isosceles trapezium with the bases a and b ($a > b$) whose lateral faces are inclined to the plane containing the base at an angle of 60°. Find the altitude of the pyramid.

Solution. Let $[SO]$ be the altitude of the pyramid (Fig. 171). From the foot of the altitude O we draw perpendiculars $[OE]\perp[AB]$, $[OF]\perp[BC]$, $[OQ]\perp[CD]$ and $[OH]\perp\perp[AD]$ and join the points E, F, Q and H to the vertex S. We thus get $[SE]\perp[AB]$, $[SF]\perp[BC]$, $[SQ]\perp[DC]$ and $[SH]\perp[AD]$ (by the theorem on three perpendiculars). The angles OES, OFS, OQS and OHS, i.e. the plane angles of the corresponding dihedral angles formed by the lateral faces of the pyramid with the plane containing the base, are equal to 60°. Hence, $\triangle OES \cong \triangle OFS \cong \triangle OHS \cong \triangle OQS$ and $|OE| = |OF| = |OQ| = |OH| = r$, where r is the radius of the circle inscribed in the trapezium $ABCD$. As is obvious, $r = |QE|/2$. To find the altitude of the pyramid $[OS]$ let us consider the triangle OSE in which only the angle E is known. Therefore we first find $|OE|$.

In the trapezium $ABCD$ (Fig. 172) we draw the altitude $[DM]$; then

$$|AM| = \frac{|AB| - |DC|}{2} = \frac{a-b}{2}.$$

Since a circle can be inscribed in the trapezium $ABCD$, we have $|AD| + |CB| = |AB| + |DC|$ or $2\,|AD| = = a + b$. Then

$$|DM| = \sqrt{|AD|^2 - |AM|^2} = \sqrt{\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} = \sqrt{ab}.$$

Thus, $|QE| = |DM| = \sqrt{ab}$. Returning to the triangle SOE, we find $|OS| = |OE| \cdot \tan 60^\circ = \frac{1}{2}\sqrt{ab} \cdot \sqrt{3} = \frac{\sqrt{3ab}}{2}$.

Problem 4. The altitude of a right circular cone is equal to the radius R of the base. Through the vertex of the cone a plane is drawn which cuts off an arc of 60° from the base circle. Find the area of the section.

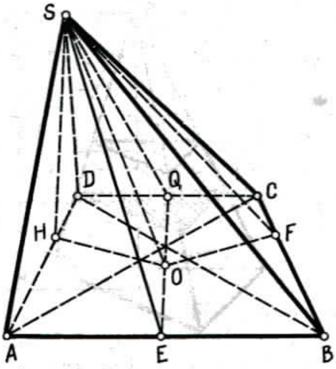


Fig. 171

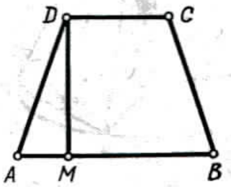


Fig. 172

Solution. The cutting plane intersects the surface of the cone along the elements $[AS]$ and $[BS]$ and the chord $[AB]$. The required area is the area of the triangle ASB (Fig. 173) in which the base $[AB]$ subtends an arc of 60°, hence, $|AB| = R$ ($\triangle AOB$ is an equilateral triangle). The height of the section $[SC]$ is the hypotenuse of the triangle SOC in which $|SO| = R$, and $|OC| = R\sqrt{3}/2$. Therefore, $|SC| = \sqrt{|SO|^2 + |CO|^2} = \frac{R\sqrt{7}}{2}$, area $ABS = \frac{1}{2}\,|AB| \cdot |SC| = \frac{R^2\sqrt{7}}{4}$.

Problem 5. The altitude of the frustum of a pyramid is equal to h and the areas of the bases are Q and q ($Q > q$). At what distance from the lower base is the section parallel to it if its area is the mean proportional of the areas of the bases?

Solution. Let us complete the drawing of the frustum to get a full pyramid and denote its altitude by H (Fig. 174).

Then

$$\frac{q}{Q}=\frac{(H-h)^2}{H^2}, \quad \text{or} \quad \frac{\sqrt{q}}{\sqrt{Q}}=\frac{H-h}{H},$$

whence $H=\dfrac{h\sqrt{Q}}{\sqrt{Q}-\sqrt{q}}$. Denoting the required distance by x

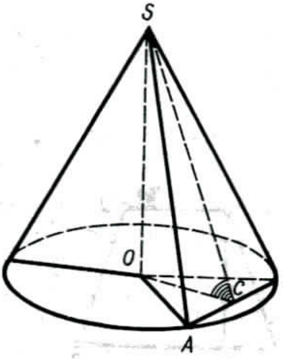


Fig. 173

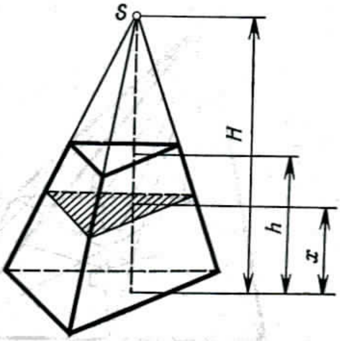


Fig. 174

and using the property of parallel sections once again, we get

$$\frac{Q}{\sqrt{Qq}}=\frac{H^2}{(H-x)^2} \quad \text{or} \quad \frac{\sqrt[4]{Q}}{\sqrt[4]{q}}=\frac{H}{H-x},$$

whence

$$x=\frac{H(\sqrt[4]{Q}-\sqrt[4]{q})}{\sqrt[4]{Q}}=\frac{h\sqrt{Q}}{\sqrt{Q}-\sqrt{q}}\cdot\frac{\sqrt[4]{Q}-\sqrt[4]{q}}{\sqrt[4]{Q}}=\frac{h\sqrt[4]{Q}}{\sqrt[4]{Q}+\sqrt[4]{q}}.$$

PROBLEMS TO CHAPTER 7

1. The edge of a regular octahedron is equal to 1 m. Determine the distance between two opposite vertices of the octahedron.

2. In a cube from one of its vertices diagonals of three adjacent faces are drawn, and their end-points are connected with straight lines. Prove that the polyhedron bounded by the four planes passing through these lines is a regular tetrahedron.

3. Join (with straight lines) the centres of each pair of adjacent faces of an octahedron and draw planes through the adjacent lines. Prove that the hexahedron thus obtained

is a cube. Compute the edge of the cube if the edge of the octahedron is equal to a.

4. In a regular tetrahedron a regular triangular prism with equal edges is inscribed so that the vertices of one of its bases are found on the lateral edges of the tetrahedron, while those of the other base lie in the plane containing the base of the tetrahedron. Determine the edge of the prism if the edge of the tetrahedron is equal to a.

5. The edge of an octahedron is equal to a. Find the distance between the centres of two of its opposite faces.

6. Find (to within 1 cm^2) the area of the diagonal section of a cube if its edge is equal to 6 dm.

7. Find the areas of the diagonal sections of a rectangular parallelepiped if its dimensions are equal to 8 cm, 6 cm, and 12 cm.

8. Find the diagonal of a parallelepiped if its dimensions are equal to: (1) 2 dm, 4 dm, 4 dm; (2) 3 m, 4 m, 12 m; (3) 40 cm, 60 cm, 120 cm.

9. Find the diagonals of a right parallelepiped the sides of whose base are equal to 10 dm and 6 dm, and one of the diagonals is equal to 8 dm. The shorter diagonal of the parallelepiped forms an angle of 60° with the plane containing its base.

10. In a right parallelepiped the sides of the base are equal to 2 cm and 5 cm; the distance between smaller sides is 4 cm; the lateral edge being equal to $2\sqrt{2}$ cm. Determine the diagonals of the parallelepiped.

11. Determine the diagonals of a right parallelepiped, each of whose edges is equal to a, the angle between the sides of the base being equal to 60°.

12. In a right parallelepiped the sides of the base (3 cm and 4 cm long) form an angle of 60°, and the lateral edge is the mean proportional between the sides of the base. Determine the diagonals of the parallelepiped.

13. The edge of a cube is equal to a. Find the distance from its vertex to the diagonal.

14. Prove that in any parallelepiped the sum of the squares of the diagonals is equal to the sum of the squares of all the edges.

15. Given in a right parallelepiped with the base $ABCD$: $|AB| = 29$ cm, $|AD| = 36$ cm, $|BD| = 25$ cm and the lateral edge is equal to 48 cm. Determine the area of the section AB_1C_1D.

16. In a right parallelepiped the acute angle of the base is equal to α, and one of the sides of the base is equal to a. The section passing through this side and the opposite edge of the upper base has an area of Q cm^2. Determine the other side of the base if the cutting plane is inclined to the plane containing the base at an angle of $90° - \alpha$.

17. In a right parallelepiped the lateral edge is equal to 1 m, the sides of the base are equal to 23 dm and 11 dm, and the diagonals of the base are to each other as 2 : 3. Determine the areas of the diagonal sections perpendicular to the plane containing the base.

18. The base of an oblique parallelepiped is a rhombus $ABCD$ in which $\widehat{BAD} = 60°$, the lateral edges are inclined to the plane of the base at an angle of 60°, and the plane AA_1C_1C is perpendicular to the plane containing the base. Prove that the area of BB_1D_1D is to the area of AA_1C_1C as 2 to 3.

19. The base of an oblique parallelepiped is a rhombus whose side is equal to 60 cm. The plane of the diagonal section passing through the longer diagonal of the base is perpendicular to the plane containing the base. The area of this section is equal to 72 dm^2. Find the smaller diagonal of the base if the lateral edge is equal to 80 cm and forms an angle of 60° with the plane of the base.

20. Given a regular triangular prism the side of whose base is equal to a. Through the mid-points A and C of two sides of the base a plane is drawn at an angle of 60° to the base which cuts the lateral edge at point B. Find the area of the section.

21. Prove that if in a regular quadrangular prism $ABCDA_1B_1C_1D_1$ the diagonals $[B_1D]$ and $[D_1B]$ are mutually perpendicular, then the diagonals $[A_1C]$ and $[B_1D]$ form an angle of 60°.

22. The base of a prism is a regular hexagon with the side a; the lateral faces are squares. Determine the diagonals of the prism and the areas of its diagonal sections perpendicular to the bases.

23. Inside a regular hexagonal prism whose faces are squares a plane is drawn through the side of the lower base and the opposite side of the upper base. Determine the area of the section if the side of the base is equal to a.

24. In a right triangular prism a plane is drawn through one of the sides of the base which intersects the opposite lateral edge and is inclined to the plane containing the base at an angle of 45°. Determine the area of the section if the area of the base is equal to Q.

25. In a right triangular prism the sides of the base are equal to 10 cm, 17 cm, and 21 cm, the altitude of the prism being equal to 18 cm. Determine the area of the section drawn through the lateral edge and the smaller altitude of the base.

26. The base of a right prism is a rhombus; the diagonals of the prism are equal to 8 cm and 5 cm, its altitude being equal to 2 cm. Find the side of the base.

27. The lateral edge of a prism is inclined to the plane of the base at an angle of 30° and is equal to 15 cm. Find the altitude of the prism.

28. In an oblique triangular prism two dihedral angles between the lateral faces are equal to $20°43'28''$ and $105°27'32''$. What is the magnitude of the third angle?

29. In an oblique triangular prism the distances between the lateral edges are equal to 37 cm, 13 cm, and 40 cm. Find the distance between the larger lateral face and the opposite lateral edge.

30. The lateral surface of a cylinder is made up from a square with the side a. Find the area of the base of the cylinder.

31. In a cylinder a plane is drawn parallel to its axis which cuts off an arc of 120° from the base circle. The axis of the cylinder is 10 cm long, its distance from the cutting plane is equal to 2 cm. Determine the area of the section.

32. The area of an axial section of a cylinder is to the area of its base as 4 to π. Find the angle between the diagonals of the axial section.

33. The altitude of a cylinder is equal to 6 dm and the radius of the base to 5 dm. The end-points of a given line segment of length 109 m lie on the base circles. Find the distance between the line segment and the axis of the cylinder.

34. Through the upper end-point of an element and at an angle of 45° to it a tangent line is drawn to a cylinder. The radius of the base circle of the cylinder is equal to 1 m, the altitude to 4 m. Determine the distances from the centres of the base circles of the cylinder to the tangent line.

35. Given the side of the base a and the altitude h, determine the slant height of a regular pyramid: (1) triangular, (2) quadrangular, (3) hexagonal.

36. Given the side of the base a and the lateral edge b, determine the altitude of a regular pyramid: (1) triangular, (2) quadrangular, (3) hexagonal.

37. The altitude of a regular quadrangular pyramid is equal to 7 cm, and the side of the base to 8 cm. Determine the lateral edge.

38. The base of a pyramid is a parallelogram whose sides are equal to 3 cm and 7 cm; one of its diagonals is equal to 6 cm. The altitude of the pyramid passes through the point of intersection of the diagonals of the base and is equal to 4 cm. Find the lateral edges.

39. Find the perimeter of the base of a regular quadrangular pyramid if its altitude is equal to 6 dm, and the slant height to 6.5 dm.

40. The base of a pyramid is an isosceles triangle in which the base is equal to 12 cm and the lateral side to 10 cm. The lateral faces form dihedral angles of 45° with the plane of the base. Find the altitude of the pyramid.

41. In a regular quadrangular pyramid a cube is inscribed so that four of its vertices are found on the lateral edges of the pyramid, the rest of them lying in the plane containing its base. The side of the base of the pyramid is equal to a and the altitude is equal to h. Find the edge of the cube.

42. The area of the base of a pyramid is equal to 507 cm^2. The section which is parallel to the base of the pyramid divides its altitude in the ratio $6:7$ (as measured from the vertex). Find the area of the section.

43. The base of a pyramid is a square. One of the edges of the pyramid is perpendicular to the plane containing the base. The longest lateral edge equal to 6 dm is inclined to the base at an angle of 45°. Find the area of the base.

44. At what distance from the vertex of a pyramid with the altitude of 12 cm must a section parallel to the base be drawn so that the area of the section is a quarter of the area of the base?

45. Prove that if a triangular pyramid is cut by a plane parallel to two of its non-intersecting edges, then the section figure is a parallelogram.

46. In a regular hexagonal pyramid the altitude is equal to h, and the side of the base to a. Find the areas of the diagonal sections.

47. In a pyramid the area of the base is equal to 150 cm^2, the area of a parallel section is 54 cm^2, the distance between them being equal to 14 cm. Find the altitude of the pyramid.

48. In a regular triangular pyramid through a side of the base a plane is drawn perpendicular to the opposite lateral edge. Find the area of the section thus obtained if the side of the base of the pyramid is equal to 1 cm, and the altitude is equal to 4 cm.

49. The altitude of a frustum of a regular quadrangular pyramid is equal to 7 cm, and the sides of the bases to 10 cm and 2 cm. Find the lateral edge of the pyramid.

50. The sides of the bases of a frustum of a regular triangular pyramid are equal to 4 dm and 1 dm, and the lateral edge to 2 dm. Find the altitude.

51. In a frustum of a regular quadrangular pyramid the altitude is equal to 63 cm, the slant height is equal to 65 cm, and the sides of the bases are to each other as 7 : 3. Find these sides.

52. In a frustum of a regular quadrangular pyramid the altitude is equal to 2 cm, and the sides of the bases to 3 cm and 5 cm. Find the diagonals of this pyramid.

53. Determine the sides of the bases of a frustum of a regular quadrangular pyramid if its altitude is equal to 7 cm, the lateral edge to 8 cm, and the diagonal to 11 cm.

54. The diagonals of a frustum of a regular quadrangular pyramid are perpendicular to the lateral edges, the side of the lower base is equal to 9 cm, and the lateral edge to 8 cm. Find the side of the upper base, the altitude of the frustum, and the distance from the point of intersection of its diagonals to the plane containing the lower base.

55. The sides of the bases of a frustum of a regular triangular pyramid are equal to 2 cm and 6 cm, the lateral face forms an angle of 60° with the larger base. Find the altitude.

56. The altitude of a frustum of a regular quadrangular pyramid is equal to 4 cm, and the diagonal is equal to 5 cm. Find the area of the diagonal section perpendicular to the bases.

57. In a frustum of a regular quadrangular pyramid the areas of the bases are equal to Q cm^2 and q cm^2, and the lateral edge forms an angle of 45° with the plane containing the lower base. Find the area of the diagonal section perpendicular to the bases.

58. In a frustum of a regular triangular pyramid the sides of the bases are equal to 8 m and 5 m, and the altitude to 3 m. Draw a secant plane through the side of the lower base and the opposite vertex of the upper base. Find the area of the section and the dihedral angle between the cutting plane and the lower base.

59. The corresponding sides of the bases in a frustum of a pyramid are to each other as 13 : 17, and the perimeter of the mid-section is equal to 45 m. Find the perimeters of the bases.

60. The areas of the bases in a frustum of a pyramid are equal to 2 m^2 and 98 m^2. Find the area of the parallel section drawn through the mid-point of the altitude.

61. The altitude of a cone is equal to 20 cm, and the radius of the base circle to 25 cm. Find the area of the section drawn through the vertex if the distance of the cutting plane from the centre of the base circle of the cone is equal to 12 cm.

62. The altitude of a cone is H, the angle between the altitude and the generatrix is equal to 60°. Find the area of the section drawn through two mutually perpendicular elements.

63. In a cone whose altitude is equal to the radius of the base circle R a plane is drawn through the vertex which cuts off an arc of 90° from the base circle. Find the area of the section obtained.

64. In a frustum of a cone the areas of the bases are equal to 1 m^2 and 49 m^2. The area of a parallel section is equal to half their sum. Into what portions is the altitude divided by this section (as measured from the vertex)?

65. In a frustum of a cone the altitude is equal to 10 cm, and the radii of the base circles to 8 cm and 18 cm. At what distance from the smaller section is the parallel section located whose area is the mean proportional between the areas of the bases?

CHAPTER 8

THE BALL

Sec. 61. A Sphere and a Ball Defined. A Sphere and a Ball Cut by a Plane

A *spherical surface* (or a *sphere*) is defined as the set of all points in space at a given positive distance $R > 0$ from a fixed point called the *centre* of the sphere.

A *ball* is a space figure bounded by a sphere; or the set of all points in space whose distance from a fixed point does not exceed a given positive number R. As is obvious, the centre of a sphere is the centre of a ball.

The *radius* of a sphere and a ball is a line segment joining their centre to any point on the sphere; the *chord* of a sphere is a line segment connecting any two of its points; a chord passing through the centre is called the *diameter*.

A ball can be generated by rotating a semi-circle about its diameter. In doing so the semi-circle describes a spherical surface (or a sphere).

Theorem. *The section of a sphere by a plane is a circle or a point.*

Given: a sphere S with the centre O, a plane α, $\rho(O, \alpha) = h$.

Prove: $S \cap \alpha$ is a circle or a point.

Proof. If the cutting plane passes through the centre of the sphere (Fig. 175), then all points of the section are located at a distance R from the point O. Consequently, the section obtained is a circle whose centre is the centre of the sphere and whose radius is equal to the radius of the sphere.

Suppose the centre of the ball O is not contained in the plane α. Dropping a perpendicular $[OA]$ from the point O onto the plane α, we have $|OA| = h$. Let us join an arbitrary point M of the section to the points O and A (Fig. 176). As is clear from the right-angled triangle OMA: $|AM| = \sqrt{R^2 - h^2}$. Hence, the distance of any point of the section $S \cap \alpha$ from the point A is a quantity constant for the given sphere and the given plane α, i.e. $S \cap \alpha$ is a circle with the centre at point A and the radius $r = \sqrt{R^2 - h^2}$ if $h < R$. But if $h = R$, then $|AM| = 0$, which means that $M = A$, i.e. the section $S \cap \alpha$ consists of one point. Thus, the theorem has been proved.

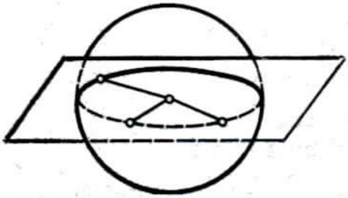


Fig. 175

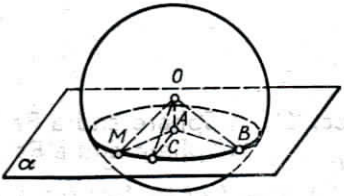


Fig. 176

As is seen from the relation $r = \sqrt{R^2 - h^2}$, the quantity r reaches its greatest value (R) when $h = 0$ (i.e. when the section passes through the centre of the sphere). That is why the circle obtained as a section figure when a ball is cut by a plane passing through its centre is called the *great circle of the ball.*

Corollary. *The sections of a ball by planes equidistant from its centre are circles of equal radii.*

Sec. 62. A Tangent Plane

Definition. A plane having only one point in common with a ball is called the *tangent plane.*

Theorem. *For a plane to be tangent to a ball (sphere) it is necessary and sufficient that it is perpendicular to the radius of the ball at its end-point lying on the sphere.*

1. Given: a plane α perpendicular to $[OA]$ (Fig. 177).
Prove: the plane α is a tangent plane.

2. Given: a tangent plane α.
Prove: the plane α is perpendicular to $[OA]$.

Proof. 1. Let us take an arbitrary point $M \neq A$ on the plane α and draw an inclined line $[MO]$. By virtue of the theorem of Sec. 32, $|MO| > |AO|$, i.e. the point M is situated outside the sphere. Since M is an arbitrary point, the obtained inequality $|MO| > |AO|$ is true for all points of the plane α, except for the point A, which is, consequently, the only point common to the sphere and to the plane, i.e. α is a tangent plane.

2. According to the definition of the tangent plane, the point of tangency A (Fig. 177) is the only point belonging both to the sphere, and to the plane ($A = \alpha \cap S$). For any other point $M \in \alpha$ we have $|OM| > |OA|$, i.e. $\rho(O, \alpha) = |OA|$. This means that (see Sec. 32) A is the foot of the perpendicular dropped from the centre O onto the plane α. The theorem is thus proved.

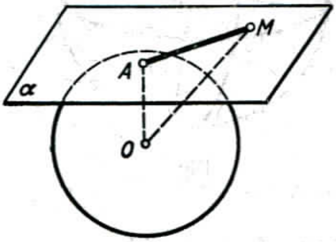


Fig. 177

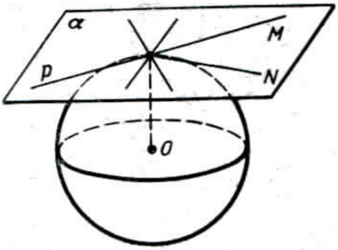


Fig. 178

A straight line having only one point in common with a sphere (a ball) is called the *tangent line* to the sphere (the ball).

It is obvious, that any straight line drawn in the tangent plane through the point of tangency A is tangent to the sphere. The following theorem is valid.

Theorem. *For a straight line to be tangent to a sphere (a ball) it is necessary and sufficient that it is perpendicular to the radius at its end-point lying on the sphere* (Fig. 178).

The proof of this theorem is left to the student.

Sec. 63. The Concept of a Spherical Triangle

Let us take on a sphere three points A, B, and C not lying in one plane with the centre of the sphere O. We then draw a great circle through each pair of the points and mark on each circumference of the circles an arc joining the points A and B, B and C, C and A (Fig. 179). The portion of the spherical surface bounded by these arcs is called a *spherical triangle* ABC. The points A, B, and C are the *vertices* of the spherical triangle, the arcs $\smile AB$, $\smile BC$ and

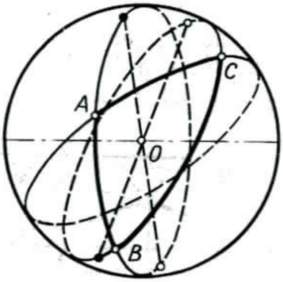


Fig. 179

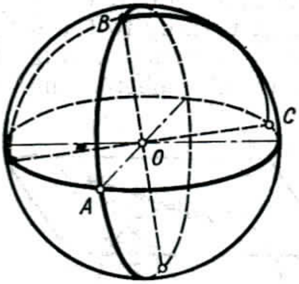


Fig. 180

$\smile AC$ being its *sides*. The lengths of these sides are also termed *spherical distances* between the points connected by them.

According to the construction of a spherical triangle, the length of its side cannot exceed half the length of the great circle of a sphere, i.e.

$$\text{length} \smile AB \leqslant \pi R,\ \text{length} \smile BC \leqslant \pi R,\ \text{length} \smile AC \leqslant \pi R.$$

The sides of a spherical triangle can also be measured in degrees. The degree measure of the side of a spherical triangle is equal to a central angle of the great circle on which this side is located (i.e. this side is a portion of the great circle). Figure 180 depicts a spherical triangle each of whose sides contains 90°, or length $\smile AB =$ length $\smile BC =$ $=$ length $\smile AC = 0.5\,\pi R$.

The angle A of a spherical triangle is the angle formed by the tangents (AT) and (AT') drawn to the arcs $\smile AB$ and $\smile AC$ at the point A of their intersection (Fig. 181). Since (AT) and (AT') are perpendicular to the radius $[OA]$ and lie respectively in the planes $AOA'B$ and $AOA'C$, the angle TAT' is the plane angle of the dihedral angle $AA'BC$, where A' is the second end-point of the diameter passing through the point A.

A spherical triangle can be obtained from a trihedral angle $OMNP$ by intersecting it with a ball whose centre coincides with the vertex O of the trihedral angle (Fig. 182).

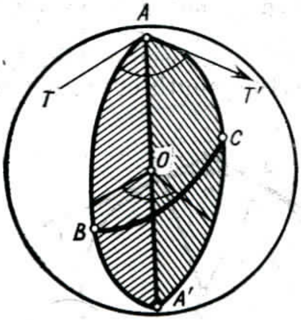


Fig. 181

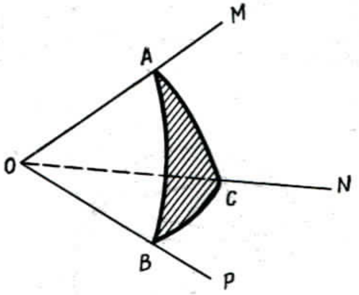


Fig. 182

Sec. 64. Worked Problems

Problem 1. Points A, B and C are located on a sphere. The rectilinear distances between them are equal to 5 cm, 7 cm, and 8 cm. The plane passing through these points is 7/3 cm distant from the centre of the sphere. Find the radius of the sphere.

Solution. When cutting the sphere, the plane ABC forms a circle in which the triangle ABC is inscribed. The radius of this circle is found by the formula $r=\frac{abc}{4S}$, where

$$S=\sqrt{p(p-a)(p-b)(p-c)}=\sqrt{10\cdot 5\cdot 3\cdot 2}=10\sqrt{3}\text{cm}^2,$$

and, hence, $r=\frac{280}{40\sqrt{3}}=\frac{7}{\sqrt{3}}$.

Now in the right-angled triangle OO_1A (Fig. 183) two sides are known: $|O_1A|=r=7/\sqrt{3}$ and $|OO_1|=7/3$. Therefore

$$R=|OA|=\sqrt{|O_1A|^2+|O_1O|^2}=\sqrt{\frac{49}{3}+\frac{49}{9}}=\frac{14}{3}\text{ cm}.$$

Problem 2. Two planes tangent to a ball form an angle of 120°. The shortest spherical distance between the points of tangency is equal to 70 cm. Find the radius of the ball.

Solution. We draw a plane through the points of tangency and the centre of the ball O. This plane intersects the ball along the great circle (Fig. 184). We then join the centre of the ball to the points of tangency A and B. Since (OA) is perpendicular to (AC), (OB) is perpendicular to (BC) and $\widehat{ACB} = 120°$, we have $\widehat{AOB} = 60°$. Consequently, the length of the arc AnB is equal to 1/6 the length of the entire

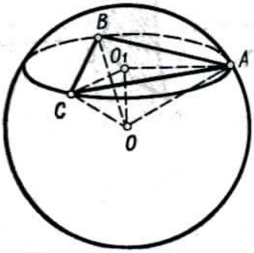


Fig. 183

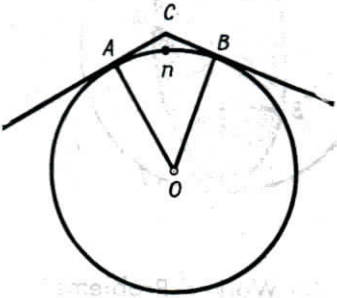


Fig. 184

circumference, i.e. $70 = (1/6)\cdot 2\pi R$, whence we find that $R = 210/\pi$. Putting $1/\pi = 0.318$, we get $R \cong 210 \times 0.318 = 66.8$ cm.

A *circular cone* is said *to be inscribed in a ball* if its vertex lies on the surface of the ball, its base being the great or a small circle of the ball (Fig. 185).

Problem 3. A circular cone is inscribed in a ball of radius R; the angle between the generating elements of the cone in an axial section is equal to α. Find the altitude, the element, and the radius of the base circle of the cone.

Solution. The section of the ball by the plane passing through the axis of the cone is the great circle in which the triangle ABS is inscribed (Fig. 185), where $[AB]$ is the diameter of the base circle of the cone. Let us extend the altitude (the axis) $[SO]$ of the cone to intersect the great circle at point E and consider the triangle ESA. In this triangle $|SE| = 2R$, $\widehat{SAE} = 90°$ and $\widehat{ASE} = \alpha/2$. Therefore, $|AS| = 2R\cdot\cos(\alpha/2)$. From the triangle AOS we now find

$$|AO| = r = 2R\cos\frac{\alpha}{2}\cdot\sin\frac{\alpha}{2} = R\sin\alpha,$$

$$|SO| = h = 2R\cos^2\frac{\alpha}{2}.$$

A *frustum of a circular cone* is said *to be inscribed in a ball* if its bases are parallel circles of the ball.

Problem 4. In a frustum of a cone the radius of the lower base circle is equal to r_1, the radius of the upper base circle is equal to r_2, and the generatrix is inclined to the

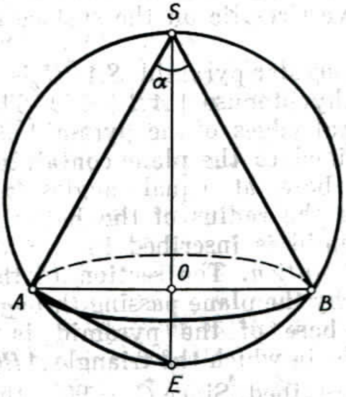


Fig. 185

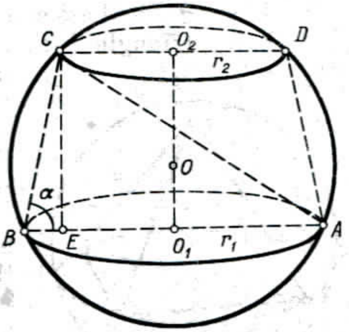


Fig. 186

plane containing the lower base at an angle α (Fig. 186). Find the radius of the ball in which the given frustum is inscribed.

Solution. The section of the ball passing through the axis of the frustum yields a great circle in which the trapezium $ABCD$ is inscribed (Fig. 186). Let us consider the triangle ABC which is also inscribed in the great circle of the ball. In this triangle the angle CBA is known and, according to the lemma of Sec. 1, $|AC| = 2R \sin \alpha$. Thus, the problem is reduced to finding $|AC|$. For this purpose let us consider the right-angled triangle ACE, in which $|AE| = r_1 + r_2$, $|CE| = |BE| \cdot \tan \alpha = (r_1 - r_2) \tan \alpha$. Consequently,

$$|AC| = \sqrt{|AE|^2 + |CE|^2} = \sqrt{(r_1+r_2)^2 + (r_1-r_2)^2 \tan^2 \alpha} =$$

$$= \frac{1}{\cos \alpha} \sqrt{(r_1+r_2)^2 \cos^2 \alpha + (r_1-r_2)^2 \sin^2 \alpha} =$$

$$= \frac{1}{\cos \alpha} \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos 2\alpha}$$

and

$$R=\frac{|AC|}{2\sin\alpha}=\frac{\sqrt{r_1^2+r_2^2+2r_1r_2\cos 2\alpha}}{\sin 2\alpha}.$$

A *pyramid* and a *frustum of a pyramid* are said *to be inscribed in a ball* if all their vertices lie on the surface of the ball.

Problem 5. The base of a triangular pyramid $SABC$ is a right-angled triangle with the hypotenuse $|AB| = c$. The lateral edges of the pyramid are inclined to the plane containing the base at equal angles (α). Find the radius of the ball the pyramid is inscribed in.

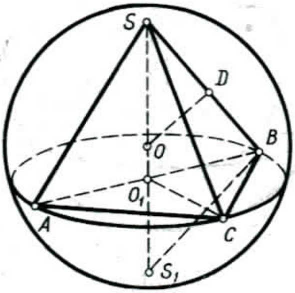


Fig. 187

Solution. The section of the ball by the plane passing through the base of the pyramid is a circle in which the triangle ABC is inscribed. Since $\widehat{C} = 90°$, the centre of this circle (O) is situated at the mid-point of the hypotenuse. The altitude of the pyramid is projected into the same point O, since, by hypothesis, the lateral edges of the pyramid are equally inclined to the plane containing the base: $\widehat{SAO_1} = \widehat{SCO_1} = \widehat{SBO_1} = \alpha$ (Fig. 187).

The centre of the circumscribed sphere is equidistant from all vertices of the pyramid, i.e. from all vertices of the base, in particular. Consequently, it is found on the perpendicular to the plane containing the base erected at the centre of the circle circumscribed about the base, i.e. at the point O. In our problem this is the altitude $[SO_1]$. (The centre of the circumscribed ball lies on the altitude of a pyramid or its extension if and only if the altitude of the pyramid passes through the centre of the circle circumscribed about the base; see Problem 11 to this chapter).

Since the centre of the circumscribed ball is equidistant from the vertices S and B, it lies in the plane passing through the mid-point of the edge $[SB]$ and perpendicular to it. We get the centre of the circumscribed ball (point O in Fig. 187) as the point of intersection of this plane and the altitude.

To find the radius R of the circumscribed circle let us consider its great circle containing the edge $[SB]$. Since $|SO_1| = \frac{c}{2}\tan\alpha$ and $|O_1S_1| = \frac{c}{2}\cdot\cot\alpha$, we have

$$2R = |SO_1| + |O_1S_1| = \frac{c}{2}(\tan\alpha + \cot\alpha) = \frac{c}{2\sin\alpha\cos\alpha},$$

whence

$$R = \frac{c}{4\sin\alpha\cos\alpha} = \frac{c}{2\sin 2\alpha}.$$

PROBLEMS TO CHAPTER 8

1. A sphere whose radius is equal to 41 dm is cut by a plane passing at a distance of 9 dm from the centre. Find the area of the section.

2. The radius of a ball is R. Through the end-point of the radius lying on the sphere a plane is drawn at an angle of 60° to the radius. Find the area of the section.

3. A town N is situated in latitude 60° North. What path is described by N during an hour due to rotation of the Earth about its axis? Put the radius of the Earth to be equal to 6000 km.

4. Three points are given on the surface of a ball. The rectilinear distances between them are equal to 6 cm, 8 cm, and 10 cm, the radius of the ball being equal to 13 cm. Find the distance from the centre of the ball to the plane passing through the three given points.

5. The radius of a ball is equal to 15 m. Outside of the ball a point A is taken at a distance of 10 m from its surface. Find the length of a circumference on the surface of the ball such that all its points are 20 m distant from the point A.

6. The radius of a ball is equal to 63 cm. A point A is located at a distance of 16 cm from the point of tangency. Find its shortest distance from the surface of the ball.

7. A hemisphere and a circular cone inscribed in it have a common base and a common altitude; a plane is drawn through the midpoint of the altitude parallel to the base. Prove that the area of the section enclosed between the lateral surface of the cone and the surface of the hemisphere is equal to half the area of the base.

8. Two equal balls of radius R are situated so that the centre of one ball lies on the surface of the other. Find the length of the line of intersection of the surfaces of these balls.

9. The radii of two balls are equal to 25 dm and 28 dm, and the distance between their centres is equal to 36 dm. Find the length of the line along which their surfaces intersect.

10. The diagonals of a rhombus are 15 cm and 20 cm long. The surface of a ball touches all sides of the rhombus. Find the distance from the centre of the ball to the plane containing the rhombus if the radius of the ball is equal to 10 cm.

11. Prove that a pyramid can be inscribed in a sphere if and only if its base can be inscribed in a circle.

12. In a ball two parallel sections are drawn on one side of its centre. The areas of the sections are equal to 49π dm^2 and 4π m^2, the distance between them being equal to 9 dm. Find the radius of the ball.

13. A regular triangular pyramid is inscribed in a sphere. The edge of its base is equal to its altitude, i.e. $a = h$. Find the radius of the sphere.

14. Find the radius of a ball circumscribed about a regular triangular pyramid whose slant heights are equal to a and form equal angles (α) with the plane containing the base.

15. A sphere is inscribed in a circular cone whose generatrix is inclined to the plane of the base at an angle α. The circle along which the conical surface touches the sphere lies in a plane situated at a distance h from the plane containing the base. Find the altitude of the cone and the radius of its base circle.

16. A sphere is inscribed in a regular quadrangular pyramid. Find the radius of the sphere if the side of the base of the pyramid is equal to a and the face angle at the vertex is equal to α.

CHAPTER 9

MEASURING VOLUMES

Sec. 65. General Properties of Volumes

In our course we shall consider the problem of measuring volumes conformably to arbitrary figures.

The volume of any solid (figure) is a positive number V describing the three-dimensional extent of a set, the following conditions being fulfilled in this respect:

1°. A cube whose edge is equal to e is adopted as unit volume, where e is unit length. This volume is denoted e^3.

2°. Congruent figures have equal volumes.

3°. If a figure T is a join of two figures F and Φ having no interior points in common, then $V(T) = V(F) + V(\Phi)$.

Two corollaries directly follow from these conditions.

Corollary 1. *If a figure F is contained in a figure Φ, then $V(F) \leqslant V(\Phi)$.*

Corollary 2. *The volume of a cube with edge p/q* (p and q integers) *is equal to* $(p/q)^3 e^3$.

It does not follow from the above-mentioned properties (1°-3°) that each figure has a volume. There are figures which have no volume. Therefore, the first problem in the theory of volume measuring is the problem of singling out figures which have volumes. It is proved that these figures are polyhedra, cylindrical and conical solids whose bases have areas, solids of revolution, etc.

Furthermore, these properties yield no method for computing the volume of a given figure. To find this method is the second problem of the theory of volume measurement.

Obviously, there is a plenty of such methods (compare with the measurement of areas), and then there arises a third problem: Do all these methods yield one and the same magnitude of the volume of a given figure?

In special courses of higher mathematics it is proved that all methods of computing the volume of a figure are equivalent, provided the condition of positiveness and properties 1° to 3° are fulfilled, which means that all the methods yield one and the same magnitude for the volume of a given figure.

One of the methods of computing volumes is the method using a three-dimensional measuring device which is a generalized two-dimensional measuring grid. Let us consider a three-dimensional figure consisting of small cubical cells whose edge is equal to $1/10^n$. We denote the join of all the cells contained in T by T_n^-, and the join of all the cells having a non-empty intersection with T by T_n^+; then $T_n^- \subset \subset T \subset T_n^+$. It is obvious that $T_n^- \subset T_1^+$ and $T_n^+ \supset T_1^-$ for any n. The quantities $V(T_n^-)$ and $V(T_n^+)$ are the lower and the upper (respectively) estimates of $V(T)$. If n increases then $T_n^- \subset T_{n+1}^-$, $T_n^+ \supset T_{n+1}^+$. Therefore, according to Corollary 1, $V(T_n^-) \leqslant V(T_{n+1}^-)$, $V(T_n^+) \geqslant V(T_{n+1}^+)$, i.e. the sequence $V(T_1^-)$, $V(T_2^-)$, ... $V(T_n^-)$, ... monotonically increases, while the sequence $V(T_1^+)$, $V(T_2^+)$, ..., $V(T_n^+)$, ... monotonically decreases and both of them are limited. Consequently, these sequences have a limit as n increases infinitely. And if it turns out that $\lim\limits_{n\to\infty} V(T_n^-) = \lim\limits_{n\to\infty} V(T_n^+)$, then their common value is taken for the volume of the solid:

$$V(T) = \lim_{n\to\infty} V(T_n^-) = \lim_{n\to\infty} V(T_n^+).$$

The volume thus determined satisfies the conditions 1°-3°. The proof of this statement is beyond our course.

Sec. 66. The Volume of a Rectangular Parallelepiped

Let us consider the general case.

Theorem. *The volume of a rectangular parallelepiped is equal to the product of its three dimensions expressed in the same units.*

Given: a rectangular parallelepiped $ABCDA_1B_1C_1D_1$, whose dimensions are a, b, and c.

Prove: $V_{\text{par}} = abc$.

Proof. Let a_n^-, b_n^-, and c_n^- denote the rational approximate (with deficit) values of the numbers a, b, and c, respectively, to within $1/10^n$ and a_n^+, b_n^+, and c_n^+ the rational approximations (with excess) with the same accuracy. We then enclose the given parallelepiped $ABCDA_1B_1C_1D_1$ with the dimensions $|AB| = a$, $|AD| = b$, $|AA_1| = c$ between the

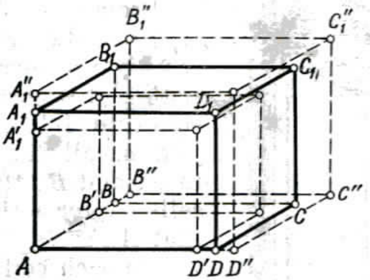


Fig. 188

rectangular parallelepipeds T_n^- and T_n^+ with the dimensions a_n^-, b_n^-, c_n^- and a_n^+, b_n^+, c_n^+, respectively, as it is shown in Fig. 188. Since $T_n^- \subset T \subset T_n^+$, then $V(T_n^-) \leqslant V(T) \leqslant V(T_n^+)$. As n increases infinitely,

$$\lim_{n \to \infty} V(T_n^-) = \lim_{n \to \infty} a_n^- b_n^- c_n^- = abc,$$

$$\lim_{n \to \infty} V(T_n^+) = \lim_{n \to \infty} a_n^+ b_n^+ c_n^+ = abc.$$

This common limit is just the volume of the given parallelepiped, i.e.

$$V(T) = abc.$$

In our proof we enclosed the given figure T between two figures, i.e. between two rectangular parallelepipeds with rational dimensions whose volumes were known.

This method is applicable in the general case. Let the figure T be contained between the figures T_n^- and T_n^+, where $V(T_n^-)$ and $V(T_n^+)$ are known. If it turns out that $\lim\limits_{n \to \infty} V(T_n^-) = \lim\limits_{n \to \infty} V(T_n^+)$, then this common limit is just $V(T)$.

Sec. 67. The Volume of a Right Cylindrical Solid

Theorem. *The volume of a right cylindrical solid is equal to the product of the area of the base by the altitude.*

Given: a right cylindrical solid T (Fig. 189); its lower and upper bases P and Q; its altitude H.

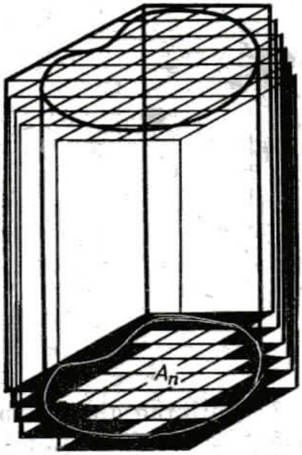


Fig. 189

Prove: $V(T) = \text{area } P \cdot H$.

Proof. Let us lay on the lower base a measuring grid consisting of squared cells with side equal to $1/10^n$. Let A_n be the join of all the cells contained in P, and B_n the join of all the cells intersecting with P, so that $A_n \subset P \subset B_n$.

On each cell of A_n and B_n as base we construct a rectangular parallelepiped with altitude H, and denote the join of all the parallelepipeds constructed on A_n by T_n^-, and those constructed on B_n by T_n^+. As is obvious, $V(T_n^-) = \text{area } A_n \cdot H$, $V(T_n^+) = \text{area } B_n \cdot H$. Since $T_n^- \subset T \subset T_n^+$, then $V(T_n^-) < V(T) < V(T_n^+)$. As n increases without bound (see Sec. 66), $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} B_n = \text{area } P$. Therefore,

$$\lim V(T_n^-) = \lim \text{area } B_n \cdot H = \text{area } P \cdot H,$$

$$\lim V(T_n^+) = \text{area } P \cdot H.$$

This common limit yields the volume of T:

$$V(T) = \text{area } P \cdot H.$$

Using this formula, we can find the volumes of any right cylinders the areas of whose bases are known: of a right prism, a right circular cylinder, etc.

Corollary 1. *The volume of a right prism is equal to the product of the area of the polygon serving as its base by its altitude:*

$$V_{\text{prism}} = S_{\text{base}} \cdot H.$$

Corollary 2. *The volume of a right circular cylinder is equal to the product of the area of the circle serving as its base by its altitude:*

$$V_{\text{cyl}} = \pi R^2 \cdot H.$$

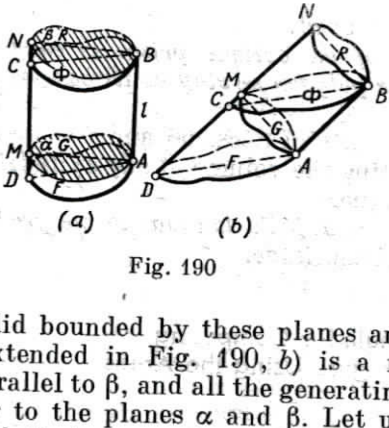
Sec. 68. The Volume of an Oblique Cylindrical Solid

Theorem. *The volume of an oblique cylindrical solid is equal to the product of the area of its base by its altitude.*

Given: an oblique cylindrical solid T; its bases F and Φ; its altitude H (Fig. 190).

Prove: $V(T) = \text{area } F \cdot H$.

Proof. Through the end-points of the element $[AB]$ we draw two planes α and β perpendicular to it (Fig. 190). The solid bounded by these planes and the lateral surface of T (extended in Fig. 190, *b*) is a right cylinder, since α is parallel to β, and all the generating elements are perpendicular to the planes α and β. Let us denote this solid by T_1, and its lower and upper bases by G and R, respectively. The solid T_1 and the solid ADM bounded by the plane figures F and G and a cylindrical surface, have G in common and together form the solid $ADNB$. According to property 2°,



(*a*) (*b*)

Fig. 190

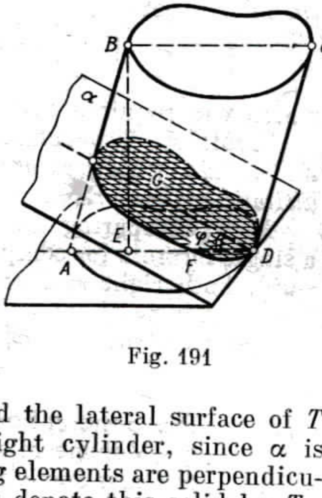


Fig. 191

$$V(ADNB) = V(T_1) + V(ADM). \qquad (1)$$

On the other hand, the solid $ADNB$ is made up of T and BCN, which is bounded by plane figures Φ and R.

Therefore,

$$V(ADNB) = V(T) + V(BCN) \tag{2}$$

The solids ADM and BCN are congruent since BCN is the image of ADM, when the latter is transferred on the vector $\overrightarrow{AB}$. By property 1°,

$$V(BCN) = V(ADM). \tag{3}$$

Comparing (1) and (2) and taking into account (3), we get: $V(T) = V(T_1) = \text{area } G \cdot |AB|$.

Let φ denote the plane angle between the planes containing F and G (Fig. 191). Since G is the orthogonal projection of F, area G = area $F \cdot \cos\varphi$ and $V(T_1) = \text{area } F \times \times \cos\varphi \cdot |AB|$, and $|AB| \cdot \cos\varphi = |BE| = H$. Therefore, we finally get

$$V(T) = V(T_1) = \text{area } F \cdot H,$$

which was required to be proved.

Corollary. *The volume of an oblique prism is equal to the product of the area of the polygon serving as its base by the altitude of the prism.*

Note. Comparing the results of Secs. 66 and 67, we get a single formula for computing the volume of any cylindrical solid, both right and oblique.

The volume of any cylindrical solid is equal to the product of the area of its base by the altitude.

Sec. 69. The General Formula for Computing the Volume of a Figure Using the Areas of Cross-Sections

Let us cut a given figure T by planes perpendicular to the axis Ox. We then denote the areas of sections thus obtained by $S(x)$, where x is the distance of a cutting plane from the origin, and $a \leqslant x \leqslant b$, where points a and b correspond to the extreme sections of T (Fig. 192). These sections can be reduced to points as well.

Theorem. *If the area of cross-sections $S(x)$ is a continuous function of x within the interval $[a, b]$, then the volume of the* *figure T can be found by the formula*

$$V = \text{volume}\, T = \int_a^b S(x)\, dx. \tag{1}$$

Proof. To simplify the deduction of this formula let us make the following additional assumptions: (1) suppose that $S(x)$ is monotonic on $[a, b]$, i.e. either increases or decreases on $[a, b]$; (2) the projections of the sections on a plane perpendicular to Ox do not intersect (Fig. 193).

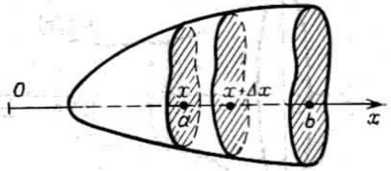


Fig. 192

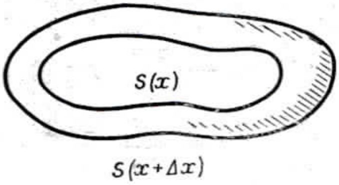


Fig. 193

Let $V(x)$ denote the volume of the portion of T contained between the sections passing through the points a and x. Then $V(x)$ is a function of x defined on $[a, b]$ and $V(a) = 0$, $V(b) = V$, where $V =$ volume T.

Let us show that $V'(x) = S(x)$ for $a \leqslant x \leqslant b$. For this purpose we give x an increment Δx $(a \leqslant x + \Delta x \leqslant b)$ and compute the increment

$$\Delta V = V(x + \Delta x) - V(x).$$

The quantity ΔV is the volume of the "layer" contained between the cutting planes passing through the points with abscissas x and $x + \Delta x$. Let us construct cylinders on these sections with a common altitude Δx. It is obvious that one of them is contained inside the "layer", and the other contains this "layer" (Fig. 194). Since the volumes of the constructed cylinders are equal to $S(x) \cdot \Delta x$ and $S(x + \Delta x) \cdot \Delta x$ (see Sec. 68), then, by the property of volumes, we have

$$S(x)\,\Delta x \leqslant \Delta V \leqslant S(x + \Delta x) \cdot \Delta x \text{ or}$$
$$S(x)\,\Delta x \geqslant \Delta V \geqslant S(x + \Delta x) \cdot \Delta x,$$

whence

$$S(x) \leqslant \frac{\Delta V}{\Delta x} \leqslant S(x+\Delta x) \qquad (2)$$

or

$$S(x) \geqslant \frac{\Delta V}{\Delta x} \geqslant S(x+\Delta x). \qquad (3)$$

Since $S(x)$ is continuous on $[a, b]$ we have $\lim\limits_{\Delta x \to 0} S(x+\Delta x) = S(x)$, therefore from the double inequality (2) or (3)

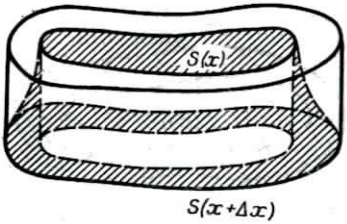


Fig. 194

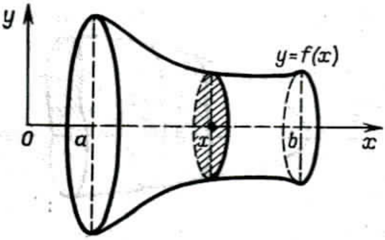


Fig. 195

it follows that $V'(x) = S(x)$ and, hence, $\int\limits_a^b S(x)\,dx = V(b) - V(a)$. Since $V(a) = 0$, $V(b) = V$, we finally get

$$V = \int\limits_a^b S(x)\,dx,$$

which was required to be proved.

Consider an important particular case when T is a solid of revolution, i.e. when it is generated by rotating a curvilinear trapezium about the axis Ox. This trapezium is bounded by a continuous curve $y = f(x)$ $(a \leqslant x \leqslant b)$, the axis Ox and the straight lines $x = a$ and $x = b$ (Fig. 195). In this case the corresponding sections are circles and $S(x) = \pi f^2(x)$. Therefore,

$$V = \int\limits_a^b S(x)\,dx = \pi \int\limits_a^b f^2(x)\,dx. \qquad (4)$$

Sec. 70. The Formulas for Computing the Volume of a Cone, a Ball and Its Parts. Simpson's Formula

Theorem 1. *The volume of any cone is equal to one third the product of its base by the altitude.*

Given: a cone T; its base Φ; its altitude $|SO| = H$ (Fig. 196).

Prove: $V_{\text{cone}} = 1/3\, S_{\text{base}} \cdot H$.

Proof. To compute the volume of a cone let us make use of the formula (1) of Sec. 69. Let us direct the axis of abscissas along the altitude $[SO]$ and measure x from the vertex of the cone: $0 \leqslant x \leqslant H$. According to the property of parallel sections of a cone (see Sec. 58), we have

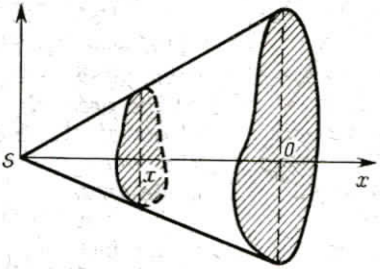


Fig. 196

$$\frac{S_{\text{base}}}{S(x)} = \frac{H^2}{x^2},$$

whence

$$S(x) = S_{\text{base}} \cdot \frac{x^2}{H^2}.$$

Hence,

$$V_{\text{cone}} = \int_0^H S(x)\,dx = \int_0^H S_{\text{base}} \cdot \frac{x^2}{H^2} dx =$$

$$= \frac{1}{H^2} \cdot S_{\text{base}} \cdot \frac{x^3}{3} \bigg|_0^H = \frac{1}{3} S_{\text{base}} \cdot H,$$

which was required to be proved.

This formula is valid for any cone, in particular for a pyramid and a circular cone. Whence we have the following corollaries.

Corollary 1. *The volume of a pyramid is equal to one third the product of the area of the polygon serving as its base by the altitude:*

$$V_{\text{pyr}} = \frac{1}{3} S_{\text{base}} \cdot H.$$

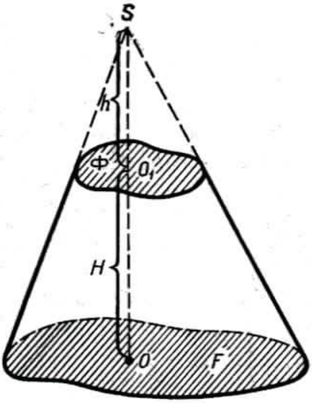


Fig. 197

Corollary 2. *The volume of a circular cone is equal to one third the product of the area of the base circle by the altitude:*

$$V_{\text{cone}} = \frac{1}{3} \pi R^2 \cdot H.$$

Theorem 2. *The volume of a frustum of a cone is equal to one third the product of its altitude by the sum of the areas of the bases of the cone and the geometric mean of these areas.*

Given: a frustum of a cone T with the bases F and Φ. $[OO_1] = H$, its altitude (Fig. 197).

Prove: $V_{\text{frustum}} = \frac{1}{3} H \ (\text{area } F + \sqrt{\text{area } F \cdot \text{area } \Phi} + \text{area } \Phi)$.

Proof. Consider the complete cone from which the given frustum was obtained. Denoting the altitude of the additional portion by h, we get

$$V_{\text{frustum}} = \frac{1}{3} \text{area } F \cdot (H + h) - \frac{1}{3} \text{area } \Phi \cdot h =$$

$$= \frac{1}{3} \text{area } F \cdot H + \frac{1}{3} (\text{area } F - \text{area } \Phi) \cdot h.$$

Let us eliminate h from the equality obtained. Since

$$\frac{\text{area } F}{\text{area } \Phi} = \frac{(H+h)^2}{h^2}, \text{ or } \frac{\sqrt{\text{area } F}}{\sqrt{\text{area } \Phi}} = \frac{H+h}{h},$$

we have

$$\frac{\sqrt{\text{area } F} - \sqrt{\text{area } \Phi}}{\sqrt{\text{area}}} = \frac{H}{h},$$

whence

$$h = \frac{\sqrt{\text{area}\,\Phi}\cdot H}{\sqrt{\text{area}\,F} - \sqrt{\text{area}\,\Phi}}$$

and

$$V_{\text{frustum}} = \frac{1}{3}\text{area}\,F\cdot H + \frac{1}{3}(\sqrt{\text{area}\,F} +$$

$$+\sqrt{\text{area}\,\Phi})\cdot H\cdot\sqrt{\text{area}\,\Phi} =$$

$$= \frac{1}{3}H(\text{area}\,F + \sqrt{\text{area}\,F\cdot\text{area}\,\Phi} + \text{area}\,\Phi),$$

which was required to be proved.

Corollary 1. *The volume of a frustum of a pyramid is computed by the formula*

$$V_{\text{frustum}} = \frac{H}{3}(\text{area}\,A + \sqrt{\text{area}\,A\cdot\text{area}\,B} + \text{area}\,B),$$

where A and B are polygons (its bases), and H is the altitude.

Corollary 2. *The volume of a frustum of a circular cone is computed by the formula*

$$V_{\text{frustum}} = \frac{H}{3}(R^2 + Rr + r^2),$$

where R and r are the radii of the base circles, and H is the altitude.

Theorem 3. *The volume of a ball of radius R is computed by the formula*

$$V_{\text{ball}} = \frac{4}{3}\pi R^3.$$

Proof. The ball is a solid of revolution, therefore, to compute its volume we make use of the formula (4) of Sec. 69. In this case $f(x) = \sqrt{R^2 - x^2}$, $-R \leqslant x \leqslant R$ (Fig. 198). Hence,

$$V_{\text{ball}} = \pi\int_{-R}^{R} f^2(x)\,dx = \pi\int_{-R}^{R}(R^2 - x^2)\,dx =$$

$$= \pi\left[R^2x - \frac{x^3}{3}\right]_{-R}^{R} = \frac{4}{3}\pi R^3,$$

which was required to be proved.

The portion of a ball contained between two parallel secant planes is called a *spherical segment* of two bases. The portion of a ball cut off by a plane is called a *spherical segment of one base* (Fig. 199).

Theorem 4. *The volume of a spherical segment of two bases is computed by the formula*

$$V_{\text{sph.segm}} = \pi R^2 H - \frac{\pi H}{3}(b^2 + ab + b^2),$$

where a and b are the distances of the cutting planes from the centre of the sphere, R is the radius of the sphere, and $H = b - a$ *is the altitude of the segment.*

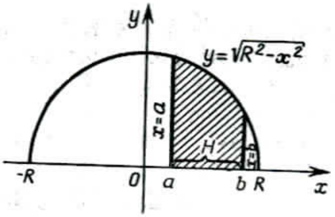


Fig. 198

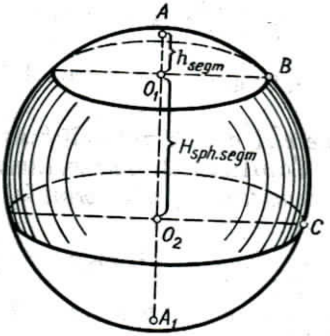


Fig. 199

Proof. A spherical segment of two bases is obtained by revolving a curvilinear trapezium bounded by an arc of a circle and straight lines $x = a$ and $x = b$ (see Fig. 198). According to formula (4) of Sec. 69, we have

$$S_{\text{sph.segm}} = \pi \int_a^b (R^2 - x^2)\,dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_a^b =$$
$$= \pi R^2 (b - a) - \frac{1}{3}(b^3 - a^3) = \pi R^2 H -$$
$$- \frac{1}{3} H (b^2 + ab + a^2),$$

which was required to be proved.

Putting $b = R$ and $a = R - H$, where H is the altitude of the spherical segment (see Fig. 199), we find

$$V_{\text{sph.segm}} = \pi R^2 H - \frac{1}{3} H \left[R^2 + (R - H) R + (R - H)^2 \right] =$$

$$= \pi R H^2 - \frac{1}{3} \pi H^3 = \pi H^2 \left(R - \frac{H}{3} \right).$$

In particular, for $H = 2R$ we get

$$V_{\text{sphere}} = \pi (2R)^2 \left(R - \frac{2}{3} R \right) = \frac{4}{3} \pi R^3.$$

All the formulas obtained in this chapter for computing the volumes are particular cases of the formula established by the English mathematician, Thomas Simpson: *for solids the areas of whose sections are quadratic functions of x, i.e.*

$$S(x) = Ax^2 + Bx + C, \qquad (1)$$

the volume is computed by the formula

$$V = \frac{H}{6} (S_1 + 4S_2 + S_3), \qquad (2)$$

where S_1 is the area of the lower section (base), S_2 is the area of the section passing through the mid-point of the altitude, and S_3 is the area of the upper section (base).

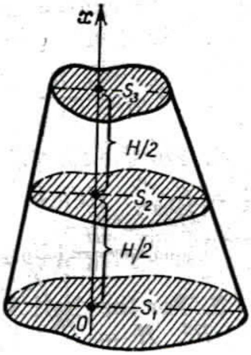


Fig. 200

Let us prove the validity of the equation (2). We know that $V = \int_a^b S(x)\,dx$ (see Sec. 69). Putting $a = 0$, $b = H$ (Fig. 200) and $S(x) = Ax^2 + Bx + C$, we find

$$V = \int_0^H (Ax^2 + Bx + C)\,dx = \left[A\frac{x^3}{3} + B\frac{x^2}{2} + Cx \right]_0^H =$$

$$= A \cdot \frac{H^3}{3} + B \cdot \frac{H^2}{2} + CH = \frac{H}{6} (2AH^2 + 3BH + 6C). \qquad (3)$$

By the condition (1) we have

$$S_1 = S(0) = C,$$

$$S_2 = S\left(\frac{H}{2}\right) = \frac{1}{4}(AH^2 + 2BH + 4C),$$

$$S_3 = S(H) = AH^2 + BH + C,$$

whence it follows that

$$S_1 + 4S_2 + S_3 = 2AH^2 + 3BH + 6C.$$

Substituting the obtained result in the right-hand side of the equation (3), we finally find

$$V = \frac{H}{6}(S_1 + 4S_2 + S_3).$$

Let us use Simpson's formula for obtaining the formulas already known to us.

1. For the cylinder $S(x) = S_{\text{base}} = \text{const}$; therefore,

$$V_{\text{cyl}} = \frac{H}{6}(S_{\text{base}} + 4S_{\text{base}} + S_{\text{base}}) = S_{\text{base}} \cdot H.$$

2. For the cone $S(x) = \frac{x^2}{H^2} \cdot S_{\text{base}}$, $S_3 = 0$, $S_2 = \frac{1}{4} S_{\text{base}}$, $S_1 = S_{\text{base}}$. Hence,

$$V_{\text{cone}} = \frac{H}{6}\left(S_{\text{base}} + 4 \cdot \frac{1}{4} \cdot S_{\text{base}} + 0\right) = \frac{1}{3} S_{\text{base}} \cdot H.$$

3. For the ball $S(x) = \pi(R^2 - x^2)$, $S_1 = S_3 = 0$; $S_2 = \pi R^2$, $H = 2R$. Thus,

$$V_{\text{ball}} = \frac{2R}{6}(0 + 4\pi R^2 + 0) = \frac{4}{3}\pi R^3.$$

Sec. 71. Worked Problems

Problem 1. A cylindrical solid is made up of two congruent circular cylinders with the altitude H and the radius of the base circle R, which intersect so that the generatrix of one cylinder coincides with the axis of the other (Fig. 201). Find the volume of this solid.

Solution. The base of the given cylindrical solid is a figure made up of two congruent circular segments. Its area is equal to twice the area of the circle less the area of their common portion which consists of two segments

(Fig. 202): $S_{\text{base}} = 2\pi R^2 - 2S_{\text{segm}}$. Since $|OD| = R/2$, we have $\widehat{AOD} = 60°$, $\widehat{AOB} = 120°$ and

$$S_{\text{sector } AOB} = \frac{\pi R^2}{3}, \quad S_{\text{segm } AO_1B} =$$

$$= \frac{\pi R^2}{3} - \frac{R^2\sqrt{3}}{4} = \frac{R^2}{12}(4\pi - 3\sqrt{3}).$$

Consequently,

$$V = S_{\text{base}} \cdot H = \left[2\pi R^2 - \frac{R^2}{6}(4\pi - 3\sqrt{3})\right] H =$$

$$= \frac{R^2 H}{6}(8\pi + 3\sqrt{3}).$$

Problem 2. The base of an oblique parallelepiped is a square with side a. One of the lateral edges forms an angle of 60° with each of the adjacent sides of the base and is equal to b (Fig. 203). Find the volume of the parallelepiped.

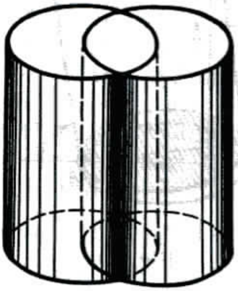


Fig. 201

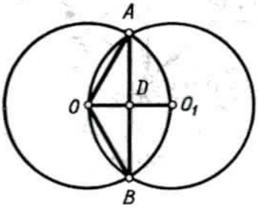


Fig. 202

Solution. Since $V = a^2H$, the problem is reduced to finding the altitude H. The edge $[AA_1]$ is inclined to the lines $[AB]$ and $[AD]$ at equal angles, consequently, the foot of the altitude $[A_1E]$ lies on the bisector of the angle BAD, i.e. on $[AC]$ (see Sec. 33).

From the point E we drop $[EF]$ perpendicular to $[AB]$, and join F to A_1; then $[A_1F]$ is perpendicular to $[AB]$. In the right-angled triangle AFA_1: $\hat{A} = 60°$, $\hat{A}_1 = 30°$, $|AA_1| = b$. Therefore, $|AF| = b/2$. Since AEF is an isosceles right-angled triangle, we have: $|EF| = |AF| = b/2$, $|AE| = \sqrt{|EF|^2 + |AF|^2} = b\sqrt{2}/2$. From the triangle AEA_1 we then have

$$|A_1E| = H = \sqrt{|AA_1|^2 - |AE|^2} = \sqrt{b^2 - \frac{2b^2}{4}} = \frac{b\sqrt{2}}{2},$$

whence $V = a^2b\sqrt{2}/2$.

Problem 3. Given an iron bar having the shape of an oblique circular cylinder with the generatrix $b = 30$ cm

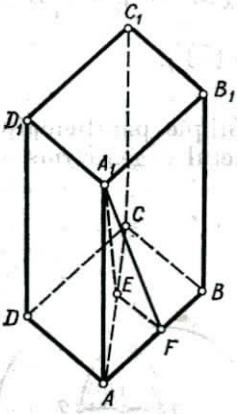


Fig. 203

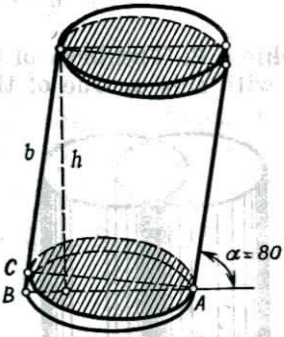


Fig. 204

and the radius of the base circle $R = 10$ cm. It is required to saw off a right cylindrical bar of the greatest possible volume. Find the ratio of the volume of the sawn portion to the volume of the given bar if its generatrix is inclined to the plane containing the base at an angle $\alpha = 80°$ (Fig. 204).

Solution. The volume of the sawn portion is equal to the difference between the volumes of the given and the obtained right cylinder. Let us first find the volume V_1 of the given cylinder. Its altitude $h = b \sin\alpha$, the area of the base $S_1 = \pi R^2$. Hence, $V_1 = \pi R^2 b \sin\alpha$. The base of the obtained cylinder is the projection of the base of the given cylinder, the angle of projection being equal to $90° - \alpha$. Therefore the area of the base of the right cylinder $S_2 =$

$= S_1 \cos(90^\circ - \alpha) = \pi R^2 \sin\alpha$, and the altitude $H = b - |CB| = b - 2R\cos\alpha$. Hence,

$$V_2 = S_2 \cdot H = \pi R^2 \sin\alpha\,(b - 2R\cos\alpha).$$

The volume of the sawn-away portion

$$V = V_1 - V_2 = \pi R^2 b \sin\alpha - \pi R^2 (b - 2R\cos\alpha)\sin\alpha = \pi R^2 \sin 2\alpha$$

and

$$\frac{V}{V_1} = \frac{\pi R^3 \sin 2\alpha}{\pi R^2 b \sin\alpha} = \frac{2R\cos\alpha}{b}.$$

Substituting the numerical values of the given quantities, we find

$$\frac{V}{V_1} = \frac{20\cos 80^\circ}{30} \approx 0.1158.$$

Problem 4. Given a piece of metal of density ρ having the shape of a triangular pyramid. It is required to turn a circular cone of the greatest possible volume with the same vertex. Find the mass of the turned-away metal if the sides of the base of the pyramid are equal to 13 cm, 14 cm, and 15 cm, and the altitude is equal to 24 cm.

Solution. The volume of the obtained circular cone (Fig. 205) is the greatest if the base of the cone is a circle inscribed in the base of the pyramid. The mass of the turned-away metal is equal to $(V_{\text{pyr}} - V_{\text{cone}})\,\rho = \frac{1}{3} H$ (area $ABC - \pi r^2)\,\rho$, where r is the radius of the circle inscribed in the triangle ABC; it is found by the formula $r = S/p$, where S is determined by Hero's formula:

$$S = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{21 \times 8 \times 7 \times 6} = 84 \text{ cm}^2;$$

$$r = \frac{84}{21} = 4 \text{ cm}.$$

Therefore,

$$M = \frac{\rho}{3} \cdot 24\,(84 - 16\pi) \approx 269.88\rho.$$

Problem 5. A sphere of radius R is inscribed in a pyramid whose base is a rhombus with an acute angle α. The lateral faces of the pyramid are inclined to the plane containing the base at equal angles φ. Find the volume of the pyramid.

Solution. The lateral faces of the pyramid are inclined to the plane of its base at equal angles, therefore the altitude of the pyramid is projected into the centre of the circle inscribed in the rhombus, i.e. into the point O of intersection of its diagonals (Fig. 206). Through this point we draw the altitude of the rhombus $[FE]$ and join E and F to the vertex

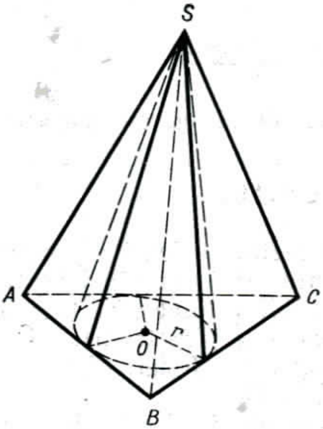


Fig. 205

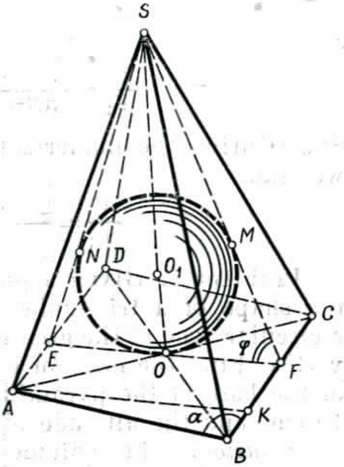


Fig. 206

S of the pyramid. By the theorem on three perpendiculars, $[SF]$ is perpendicular to $[BC]$ and $[SE]$ is perpendicular to $[AD]$.

The points at which the spherical surface touches the lateral faces of the pyramid lie on the slant heights. The volume of the pyramid

$$V=\frac{1}{3}S_{\text{base}}\cdot|SO|=\frac{1}{3}|AD|\cdot|EF|\cdot|SO|.$$

The altitude $|SO|$ is found from the triangle SEF in which the radius R of the incircle and the angles at the base φ are known (Fig. 207):

$$|OF|=|OO_1|\cdot\cot(\varphi/2)=R\cdot\cot(\varphi/2),$$
$$|EF|=2R\cot(\varphi/2),$$
$$|SO|=|OF|\cdot\tan\varphi=R\cot(\varphi/2)\tan\varphi.$$

We now find the side and then the area of the rhombus. From the right-angled triangle ABK ($[AK] \perp [BC]$, $[AK] = [EF]$) we have

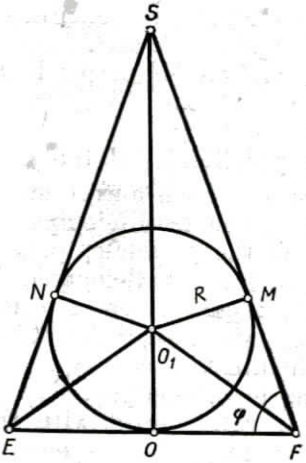


Fig. 207

$$|AD| = |AB| = \frac{|AK|}{\sin\alpha} = \frac{2R\cot(\varphi/2)}{\sin\alpha}.$$

Hence,

$$S_{\text{base}} = |AD| \cdot |EF| = \frac{4R^2\cot^2(\varphi/2)}{\sin\alpha}$$

and

$$V = \frac{1}{3} R\cot(\varphi/2)\tan\varphi\,\frac{4R^2\cot^2(\varphi/2)}{\sin\alpha} = 4R^3\,\frac{\cot^3(\varphi/2)\tan\varphi}{\sin\alpha}.$$

PROBLEMS TO CHAPTER 9

1. By how many cubic centimetres will the volume of a cube increase if its edge equal to 1 m is increased by 1 cm?

2. Three brass cubes with edges 3 cm, 4 cm, and 5 cm are smelted into one cube. Find the length of the edge of the obtained cube.

3. If each edge of a cube is increased by 2 cm, then its volume will increase by 98 cm³. Determine the edge.

4. The areas of three faces of a rectangular parallelepiped are 2 m², 3 m², and 6 m². Find its volume.

5. The diagonal of a rectangular parallelepiped is equal to l and forms an angle of 30° with one face and 45° with the other. Find its volume.

6. The base of a right parallelepiped is a rhombus. The area of the base is equal to 60 cm^2. The areas of the diagonal sections of the parallelepiped perpendicular to the base are equal to 72 cm^2 and 60 cm^2. Find the volume of the parallelepiped.

7. In a right parallelepiped the sides of the base are equal to $2\sqrt{2}$ cm and 5 cm and form an angle of 45°; the smaller diagonal of the parallelepiped is equal to 7 cm. Find the volume of the parallelepiped.

8. The base of the parallelepiped is a rhombus; the diagonal sections are perpendicular to the plane containing the base, their areas are 100 cm^2 and 105 cm^2, and the length of the line of their intersection is equal to 10 cm. Determine the volume of the parallelepiped.

9. In a right parallelepiped with the base $ABCD$ the edge $[AB]$ is 50 cm long; the perpendicular $[B_1E]$ dropped from the vertex B_1 onto $[AD]$ is equal to 41 cm and divides $[AD]$ into the segments $[AE] = 30$ cm and $|ED| = 18$ cm. Find the volume of the parallelepiped.

10. The diagonal of a regular quadrangular prism is equal to 3.5 m, and diagonal of the lateral face is equal to 2.5 m. Find the volume of the prism.

11. In a regular hexagonal prism the area of the greatest diagonal section perpendicular to the base is equal to 4 m^2, the distance between two opposite faces being equal to 2 m. Find the volume of the prism.

12. The altitude of a right triangular prism is equal to 5 m, its volume to 24 m^3, and the areas of the lateral faces are to one another as $17 : 17 : 16$. Find the sides of the base.

13. The area of the base of a right triangular prism is equal to 4 cm^2 and the areas of its lateral faces are equal to 9 cm^2, 10 cm^2, and 17 cm^2. Find the volume of the prism.

14. The base of a right prism is a trapezium in which the parallel sides $|AD| = 39$ cm, $|BC| = 22$ cm; and the lateral sides $|AB| = 26$ cm and $|CD| = 25$ cm. The area of the section ACC_1A_1 is equal to 400 cm^2. Find the volume of the prism.

15. The base of an oblique parallelepiped is a parallelogram $ABCD$ in which $|AB| = 3$ dm, $|AD| = 7$ dm,

$|BD| = 6$ dm. The diagonal section AA_1C_1C is perpendicular to the plane containing the base and is equal to 1 m^2. Find the volume of the parallelepiped.

16. The base of an oblique parallelepiped is a rhombus with side a and acute angle of 60°. The edge $[AA_1]$ is also equal to a and forms an angle of 45° with each of the edges $[AB]$ and $[AD]$. Find the volume of the parallelepiped.

17. The base of a prism is a regular triangle with side a. The vertex A_1 is projected into the centre of the base, and the edge $[AA_1]$ is inclined to the base at an angle of 45°. Find the volume of the prism.

18. In an oblique triangular prism the area of one of the lateral faces is equal to m^2, its distance from the opposite edge being equal to $2a$. Find the volume of the prism.

19. A regular triangular prism is inscribed in a cylinder, and a cylinder is inscribed in the prism. Find the ratio of the volumes of the cylinders.

20. A rectangular sheet of tin-plate whose size is 1.6 m by 0.8 m is rolled up into a tube in two different ways. Find the ratio of the volumes of the tubes thus obtained.

21. Find the ratio of the volumes of a tetrahedron and an octahedron if the mid-points of the edges of the tetrahedron serve as the vertices of the octahedron.

22. Find the volume of a right circular cylinder inscribed in a regular hexagonal prism each of whose edges is equal to a.

23. A pyramid A_1AFE is cut off from a cube $ABCDA_1B_1C_1D_1$ whose edge is equal to 24 cm; F belongs to $[AB]$ and E to $[AD]$ so that $|AE| = |AF| = 12$ cm. Find the ratio of the volume of the cut-off portion to the volume of the remaining portion.

24. Find the volume of a triangular pyramid whose lateral edges are mutually perpendicular and are equal to 10 cm, 15 cm, and 9 cm.

25. The base of a pyramid is a rectangle with sides 9 m and 12 m. Each of the lateral edges is equal to 12.5 m. Find the volume of the pyramid.

26. The base of a pyramid $SABC$ is a triangle ABC in which $|AB| = 15$ cm, $|BC| = 27$ cm, and $|AC| = 18$ cm. The faces SAB and SAC are perpendicular to the plane (ABC), and the face SBC is inclined to it at an angle of 45°. Find the volume of the pyramid and the area of the face BSC.

27. The volume of a circular cone is equal to V and the radius of the base circle is equal to R. Find the area of the axial section of the cone.

28. The radius of the base of a circular cone is equal to R. Find the radius of the section which is parallel to the base and halves the volume of the cone.

29. The liquid contained in a conical vessel whose altitude is 0.18 m and the diameter of the base circle is equal to 0.24 m is poured from this vessel into a cylindrical vessel the diameter of whose base is equal to 0.10 m. What will be the level of the liquid in the cylindrical vessel?

30. In a frustum of a triangular pyramid a plane is drawn through the side of the upper base parallel to the opposite lateral edge. In what ratio is the volume of the frustum divided if the parallel sides of its bases are to each other as $1:2$?

31. A lateral edge of a frustum of a regular pyramid is equal to 4 m, and is inclined to the plane containing the base at an angle of 60°. The radius of the circle circumscribed about the smaller base is equal to 1 m. Find the volume of the frustum.

32. The volume of a frustum of a pyramid is equal to 1720 m^3, its altitude to 20 m, its like sides are to each other as $5:8$. Find the areas of the bases.

33. A frustum of a regular quadrangular pyramid is cut from two opposite sides by two planes drawn through the end-points of the diagonal of the upper base and perpendicular to this diagonal. The sides of the bases of the pyramid are a and b, and the altitude is h. Find the volume of the remaining portion.

34. Through the point of intersection of the diagonals of a frustum of a regular quadrangular pyramid a plane is drawn parallel to its bases. The sides of the bases of the frustum are equal to 6 m, and 3 m, the altitude being equal to 9 m. Find the volume of each of the frustums obtained.

35. A frustum of a circular cone and a cone of the same altitude are equivalent. What is the radius of the base circle of the cone if the radii of the bases of the frustum are equal to 3 cm and 5 cm?

36. A frustum of a circular cone with radii of the bases of 4 cm and 22 cm is equivalent to a circular cylinder of the same altitude. Find the radius of the base of this cylinder.

37. A right-angled trapezium rotates about the lateral side which forms right angles with the bases. The area of the trapezium is 68 cm^2, its bases being equal to 10 cm and 7 cm. Find the volume of the solid of revolution.

38. Find the volume of the figure generated by a regular hexagon rotating about its side equal to a.

39. The base of a pyramid is an isosceles right-angled triangle ABC. The lateral face adjacent to the hypotenuse of the base is perpendicular to the plane containing the base. Inscribed in the pyramid is a circular cone whose volume is V. Find the volume of the pyramid.

Hint: $V_{\text{pyr}} : V_{\text{cone}} = \text{area } ABC : \pi r^2$, where r is the radius of the circle inscribed in the base of the pyramid.

40. In a circular cone through its altitude two planes are drawn at an angle of 30° to each other. The altitude of the cone is equal to H, the radius of its base circle is equal to R. Find the volumes of the portions into which the cone is divided by the planes.

41. Inscribed in a sphere is a regular triangular pyramid the side of whose base and the altitude are equal to a. Find the volume of the sphere.

42. How many small balls of diameter 12 mm can be cast from 50 kg of lead whose density is 11 400 kg/m^3?

43. Find the ratio of the volumes of a cylinder, a ball, and a cone if the diameters of the bases of the cylinder and the cone, and their altitude are equal to the diameter of the ball.

44. A vessel has a shape of an overturned circular cone whose axial section is an equilateral triangle. An iron ball of radius R is placed in it and the vessel is then filled with water so that its surface touches the submerged ball. What will the water level be after the ball is taken out of the vessel?

45. Given a ball. A plane perpendicular to its diameter divides it into two segments: 3 cm and 9 cm long. Into what portions is the volume of the ball divided?

46. Two equal balls are situated so that the centre of one ball lies on the surface of the other. How is the volume of the common portion to the volume of the entire ball?

47. The diameter of a ball is equal to 20 cm. Find the volume of a spherical segment of one base whose altitude is equal to 6 cm.

48. At what distance from the centre must a ball of radius R be intersected by a plane so that the volume of the smaller spherical segment is to the volume of the corresponding spherical sector as $5:8$?

49. At what distance from the centre must a ball be cut by a plane so that the volume of the cylinder constructed on the section figure as base and with the altitude equal to the altitude of the spherical segment of one base is equal to the volume of the segment?

50. Determine the volume of a spherical sector if the radius of its base circle is equal to 60 cm, and the radius of the sphere is 75 cm.

51. The radii of the bases of a spherical segment of two bases are equal to 3 m and 4 m, and the radius of the ball to 5 m. Find the volume of the spherical segment (consider two cases).

52. The angle between the axis of a cone and its generatrix is equal to α. Determine the ratio of the volume of the cone to the volume of the sphere circumscribed about it.

53. Inscribed in a cone is a sphere which touches the lateral surface of the cone along a circle of radius r. The generatrix of the cone is seen from the centre of the sphere at an angle α $(90^\circ < \alpha < 135^\circ)$. Find the volume of the cone.

54. Determine the angle between the generatrix and the plane containing the lower base of a frustum of a circular cone if the volume of the frustum is three times the volume of the sphere inscribed in it.

55. From an end-point of the diameter of a ball a chord is drawn so that the surface generated by its rotation about the diameter divides the volume of the ball into two equal parts. Find the angle between the chord and the diameter.

CHAPTER 10

THE AREA OF SURFACES

Sec. 72. The Area of the Surface of a Polyhedron

Definition. The *area of the surface of a polyhedron* is the sum of the areas of all its faces.

Processing from this definition, let us derive the formulas for computing the areas of the surfaces of a prism and a pyramid.

1. The Area of the Surface of a Prism. The prism is bounded by two congruent polygons serving as its bases and parallelograms forming its lateral surface. Consequently, the area of the lateral surface of the prism is equal to the sum of the areas of the parallelograms serving as its lateral faces. The area of the complete surface of the prism is equal to the sum of the areas of its lateral surface and the areas of its bases.

Theorem. *The area of the lateral surface of a prism is equal to the product of the length of its lateral edge by the perimeter of the perpendicular section of a prism.*

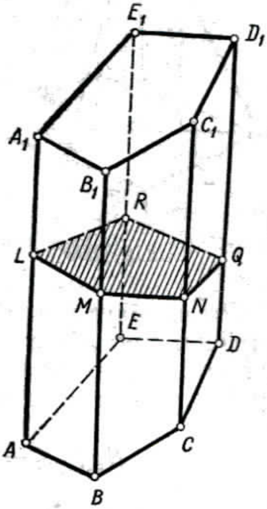


Fig. 208

Given: a prism $ABCDEA_1B_1C_1D_1E_1$, a perpendicular section $LMNQR$ (Fig. 208), $|AA_1| = l$, $|LM| + |MN| + |NQ| + |QR| + |RL| = P$.

Prove: $S_{\text{lat. pr}} = Pl$.

Proof. Let us take the lateral edges of the prism for the bases of the parallelograms belonging to the lateral surface of the prism. Then the side of the perpendicular section lying in the plane of each face of the prism is the altitude of the corresponding parallelogram. Consequently,

$$S_{\text{lat.pr}} = |AA_1| \cdot |LM| + |BB_1| \cdot |MN| + |CC_1| \cdot |NQ| + \\ + |DD_1| \cdot |QR| + |EE_1| \cdot |RL| = l \cdot (|LM| + |MN| + \\ + |NQ| + |QR| + |RL|) = lP.$$

Corollary. *The lateral surface of a right prism is equal to the product of the perimeter of its base by the lateral edge.*

2. The Area of the Surface of a Pyramid. The surface of a pyramid consists of the polygon of its base and the triangles which form its lateral surface. Consequently, the area of the lateral surface of the pyramid is equal to the sum of the areas of all the triangles forming its lateral surface.

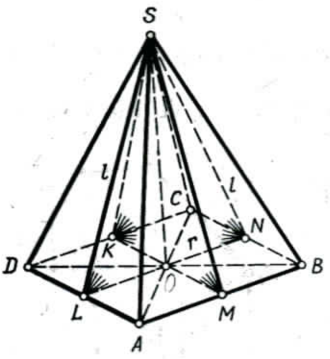


Fig. 209

Theorem 1. *The area of the lateral surface of a regular pyramid is equal to the product of the semiperimeter of its base by the length of the slant height.*

Given: a regular pyramid $SABCD$ in which $|AB| = a$, and $|SM| = l$ is the length of the slant height.

Prove: $S_{\text{lat.pyr}} = 1/2\ Pl$.

Proof. Since the triangles forming the lateral surface of a regular pyramid are congruent, the lateral surface of the pyramid is equal to the area of one such triangle multiplied by the number n of the triangles, i.e.

$$S_{\text{lat.pyr}} = n \cdot \frac{al}{2} = \frac{l}{2} \cdot na = \frac{1}{2} Pl.$$

This formula remains true for any pyramid whose lateral faces have congruent heights (see Problem of Sec. 31).

Indeed, in this case (Fig. 209)

$$S_{\text{lat.pyr}} = \frac{1}{2}|AB| \cdot l + \frac{1}{2}|BC| \cdot l + \frac{1}{2}|CD| \cdot l + \\ + \frac{1}{2}|DA| \cdot l = \frac{1}{2} Pl.$$

Theorem 2. *The area of the lateral surface of a regular pyramid is equal to the area of its base divided by* $\cos\varphi$, *where* φ *is the angle at which the lateral faces are inclined to the plane containing the base.*

Given: a regular pyramid $SABCD$ in which $[SO]$ is the altitude, $[SM]$ is the slant height, and $\widehat{SMO} = \varphi$.

Prove: $S_{\text{lat. pyr}} = \frac{S_{\text{base}}}{\cos\varphi}$.

Proof. The projections of the slant heights are the segments $[OM]$, $[ON]$, $[OK]$, and $[OL]$. The length of each of them is equal to the radius r of the circle inscribed in the base of the pyramid (Fig 209). According to the preceding theorem, we have: $S_{\text{lat. pyr}} = 1/2\, Pl$. Since $l = r/\cos\varphi$, $S_{\text{lat.pyr}} = Pr/2\cos\varphi$, whence, taking into account that $Pr/2 = S_{\text{base}}$, we finally have

$$S_{\text{lat.pyr}} = \frac{S_{\text{base}}}{\cos\varphi}. \qquad (1)$$

Note. Theorem 2 is valid not only for a regular pyramid, but also for any pyramid whose lateral faces have equal heights. In this case the lateral faces are inclined to the plane of the base at equal angles, and the reasoning of Theorem 2 thus remains unchanged.

3. The Area of the Surface of a Frustum of a Pyramid. The surface of a frustum of a pyramid consists of two polygons serving as its bases and the lateral surface made up of trapeziums.

Theorem. *The area of the lateral surface of a frustum of a regular pyramid is equal to the product of half-sum of the perimeters of its bases by the slant height.*

Given: a frustum of a regular pyramid $ABCDEA_1B_1C_1D_1E_1$ in which l is the slant height, P and Q are the perimeters of the upper and lower bases, respectively.

Prove: $S_{\text{lat. fr. pyr}} = \frac{P+Q}{2} \cdot l$.

Proof. All faces of the lateral surface of the frustum of a regular pyramid are congruent isosceles trapeziums (Fig. 210), the area of each of them being equal to $\frac{a+b}{2} \cdot l$. Hence,

$$S_{\text{lat.fr.pyr}} = n \cdot \frac{a+b}{2} = \frac{P+Q}{2} \cdot l.$$

As is obvious, the area of the lateral surface of the frustum is equal to the difference between the area of the lateral surface of a complete pyramid and the area of the lateral surface of the upper portion (Fig. 211). Therefore, for a frustum of a regular pyramid, as well as for a frustum of a pyramid whose faces are equally inclined to the planes of the bases, we can obtain a formula analogous to formula (1):

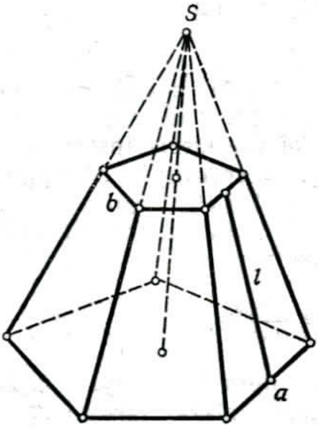


Fig. 210

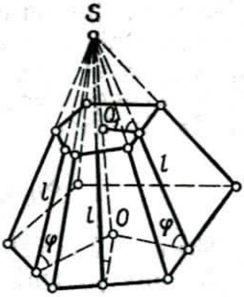


Fig. 211

$$S_{\text{lat.fr.pyr}} = \frac{S_l - S_u}{\cos\varphi}.$$

Indeed, if S_l and S_u are the areas of the bases of the frustum of a pyramid, then

$$S_{\text{lat.fr.pyr}} = \frac{S_l}{\cos\varphi} - \frac{S_u}{\cos\varphi} = \frac{S_l - S_u}{\cos\varphi}.$$

Sec. 73. The Area of an Arbitrary Surface

In computing the areas of the surfaces of polygons, prisms and pyramids in particular, we suffered no additional difficulties, since these surfaces consist of a join of polygons, i.e. of plane figures whose areas can be easily computed. Cylindrical and conical surfaces with curvilinear directrices, as also spherical surfaces, belong to curved surfaces, no portion of which (excepting some lines) can be brought in coincidence with a plane. Hence, when measuring the "area" of such a surface we are unable to compare it with a plane unit area. Therefore it is necessary to define the term "the area of a surface".

Let us consider the following example.

It is necessary to compute the quantity, i.e. the volume V, of varnish required for varnishing the floor of area S. If the thickness of the varnish coat is equal to ε, then $V(\varepsilon) = S\varepsilon$. It follows from this equation that $S = \frac{V(\varepsilon)}{(\varepsilon)}$, this formula being an exact one. That means: given the volume of a cylindrical segment and its altitude or thickness we can find the area of the base.

Let us now varnish a curved surface. Then the ratio $\frac{V(\varepsilon)}{(\varepsilon)}$ prompts us an idea as to the magnitude of the area of the surface to be varnished. The thinner the varnish coat, the more exact is this idea.

Consequently, we take $\lim\limits_{\varepsilon \to 0} \frac{V(\varepsilon)}{(\varepsilon)}$ for the area of the surface. To compute this limit we have, firstly, to specify ε more exactly and, secondly, to know the volume of the varnish coat $V(\varepsilon)$ as a function of ε. Then

$$S_{\text{surf}} = \lim_{\varepsilon \to 0} \frac{V(\varepsilon)}{\varepsilon} = \frac{dV}{d\varepsilon}.$$

Let us now consider how this method is used in computing the areas of surfaces of some solids of revolution.

Sec. 74. The Area of the Surface of a Right Circular Cylinder, a Circular Cone, and a Ball

Theorem 1. *The area of the lateral surface of a right circular cylinder is equal to the product of the length of the base circle by the altitude (generatrix) of the cylinder.*

Given: a right circular cylinder T in which $|O_1O| = H$ is the altitude, and $|OA| = R$ is the radius of the base circle (Fig. 212).

Prove: $S_{\text{lat. cyl}} = 2\pi RH$.

Proof. Let us denote the volume of the given cylinder by $V(R)$. Then the volume of the cylinder with the same altitude H and the radius of the base $R + \Delta R$ is equal to $V(R + \Delta R)$. We then denote the volume of the "layer"

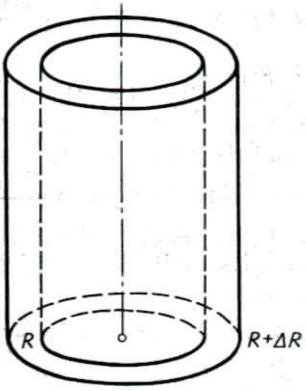


Fig. 212

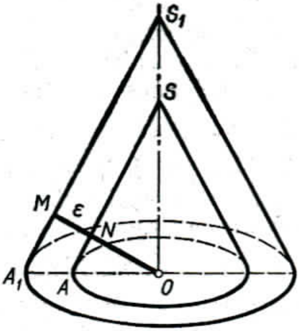


Fig. 213

contained between these cylinders by $V(\varepsilon)$, where $\varepsilon = \Delta R$ (Fig. 212).

Obviously, $V(\varepsilon) = V(\Delta R) = V(R + \Delta R) - V(R)$ and, hence,

$$S_{\text{lat.cyl}} = \lim_{\Delta R \to 0} \frac{V(R+\Delta R) - V(R)}{\Delta R} = \frac{dV}{dR}.$$

Since $V(R) = \pi R^2 H$, we have

$$S_{\text{lat.cyl}} = \frac{d}{dR}(\pi R^2 H) = 2\pi R H.$$

Theorem 2. *The area of the lateral surface of a circular cone is equal to the product of half the length of the base circle by the length of the generatrix.*

Given: a circular cone T, in which $|SA| = l$ is the generatrix, and $|OA| = R$ is the radius of the base circle (Fig. 213).

Prove: $S_{\text{lat. cone}} = \pi R l$.

Proof. We denote the volume of the given cone by V_1. Let then $V(\varepsilon)$ denote the volume of the "layer" contained between the given cone and the cone generated by rotating the triangle OA_1S_1 (where $[A_1S_1]$ is parallel to $[AS]$) about the axis OS_1 (Fig. 213). We denote the volume of this cone by V_2. Let us draw in the axial section of the cones $[OM]$ perpendicular to $[A_1S_1]$ and put the distance between the generatrices $[AS]$ and $[A_1S_1]$ to be equal to: $\varepsilon = |OM| - |ON| = (n + \Delta n) - n = \Delta n$. Obviously, $V(\varepsilon) = V_2 - V_1 = V(n + \Delta n) - V(n)$. Then

$$S_{\text{lat.cone}} = \lim_{\varepsilon \to 0} \frac{V(\varepsilon)}{\varepsilon} = \lim_{\Delta n \to 0} \frac{V(n+\Delta n) - V(n)}{\Delta n} = \frac{dV}{dn}.$$

Let us express the volume of the cone T in terms of n. Denoting $\widehat{OAS} = \varphi$, we have

$$R = |AO| = \frac{|ON|}{\sin\varphi} = \frac{n}{\sin\varphi}\ (\triangle ONA),$$

$$H = |SO| = \frac{n}{\cos\varphi}\ (\triangle ONS).$$

Then $V(n) = \frac{1}{3}\frac{\pi n^3}{\sin^2\varphi\cos\varphi}$ and

$$S_{\text{lat.cone}} = \frac{d}{dn}\left(\frac{1}{3}\frac{\pi n^3}{\sin^2\varphi\cos\varphi}\right) = \frac{\pi n^2}{\sin^2\varphi\cos\varphi}.$$

Taking into account that $\frac{n}{\sin\varphi} = R$ and $\frac{n}{\sin\varphi\cos\varphi} = \frac{|AO|}{\cos\varphi} = |AS| = l$, we get

$$S_{\text{lat.cone}} = \pi R l.$$

Theorem 3. *The area of the lateral surface of a frustum of a cone is equal to the product of the half-sum of the lengths of its bases by the length of the generatrix.*

Given: a frustum of a cone T, R and r are the radii of its bases $(R > r)$, and the length of the generatrix l.

Prove: $S_{\text{lat. fr. cone}} = \pi(R + r)\,l$.

Proof. Let us complete the cone and denote the length of the generatrix of the newly constructed portion by x, i.e. $x = |MS|$ (Fig. 214). Then

$$S_{\text{lat.fr.cone}} = \pi R(l + x) - \pi r x = \pi R l + \pi(R - r)\,x. \qquad (1)$$

To determine x let us consider the proportion $\frac{R}{r} = \frac{l+x}{x}$, whence $Rx = rl + rx$, or $(R - r)\,x = rl$. Returning to equation (1), we find

$$S_{\text{lat.fr.cone}} = \pi R l + \pi r l = \pi(R + r)\,l.$$

Note. The three formulas which we obtained for determining the areas of the lateral surfaces of a right circular cylinder, a circular cone, and a frustum of a circular cone can be reduced to one general formula, taking into consideration that R in the cylinder, $R/2$ in the cone, and $(R+r)/2$ in the frustum of a cone are equal to the radii of the parallel sections passing through the midpoint of the altitude of the corresponding solid. Then

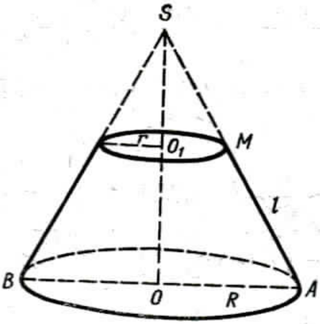


Fig. 214

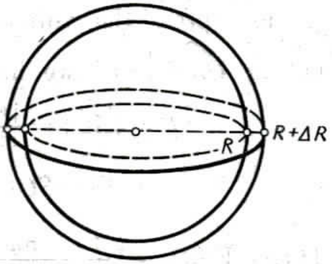


Fig. 215

$$S_{\text{lat.cyl}} = 2\pi R \cdot l, \quad S_{\text{lat.cone}} = 2\pi \frac{R}{2} \cdot l,$$

$$S_{\text{lat.fr.cone}} = 2\pi \frac{R+r}{2} \cdot l,$$

i.e. *the area of the lateral surface of each of the three solids (a right circular cylinder, a circular cone, and a frustum of a circular cone) is equal to the product of the length of the circle of the mid-section by the length of the generatrix.*

Theorem 4. *The area of the surface of a ball is equal to the product of the length of the circumference of the great circle by the length of the diameter.*

Given: a ball T and its radius R.

Prove: $S_{\text{ball}} = 2\pi R \cdot 2R = 4\pi R^2$.

Proof. Let $V(\varepsilon)$ denote the volume of the "layer" contained between the given ball and a concentric one whose radius is equal to $R + \Delta R$ (Fig. 215). Putting $\varepsilon = \Delta R$, we have

$$V(\varepsilon) = V(\Delta R) = V(R+\Delta R) - V(R);$$

$$S_{\text{ball}} = \lim_{\Delta R \to 0} \frac{V(R+\Delta R) - V(R)}{\Delta R} = \frac{dV}{dR}.$$

Since $V(R) = \frac{4}{3}\pi R^3$, we get

$$S_{\text{ball}} = \frac{d}{dR}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2.$$

Note. This formula is a particular case of a more general formula for computing the area of the surface of a zone of a sphere $S_{\text{zone}} = 2\pi R \cdot H$, where H is the altitude of the zone. We omit here the deduction of this formula.

Sec. 75. Worked Problems

Problem 1. Find the ratio of the area of the surface of a cube to the area of the surface of an octahedron inscribed in it.

Solution. Suppose the edge of the cube is equal to a, then $S_{\text{cube}} = 6a^2$. The surface of the octahedron consists of eight regular triangles whose sides join the centres of two adjacent faces of the cube (Fig. 216) and are equal. Hence,

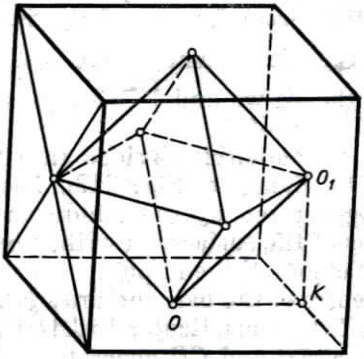


Fig. 216

$$\sqrt{|OK|^2 + |O_1K|^2} = \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \frac{a\sqrt{2}}{2}.$$ Since $S_{\text{oct}} = 8 \cdot \left(\frac{a\sqrt{2}}{2}\right)^2 \cdot \frac{\sqrt{3}}{4}$, we have $S_{\text{oct}} = a^3\sqrt{3}$ and $\frac{S_{\text{cube}}}{S_{\text{oct}}} = 2\sqrt{3}$.

Problem 2. The base of a pyramid is a triangle with the sides of 13 cm, 14 cm, and 15 cm. The lateral faces of the pyramid are inclined to the plane containing the base at an angle of 60°. Find the area of the total surface of the pyramid.

Solution. According to the note of Sec. 72,

$$S_{\text{lat.pyr}} = \frac{S_{\text{base}}}{\cos \alpha};$$

therefore

$$S_{\text{pyr}} = \frac{S_{\text{base}}}{\cos 60^\circ} + S_{\text{base}},$$

and

$$S_{\text{base}} = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84\ \text{cm}^2.$$

Hence, $S_{\text{complete pyr}} = 3S_{\text{base}} = 84 \cdot 3 = 252\ \text{cm}^2$.

Problem 3. The lateral surface of a cone is equal to Q cm², and the angle of its development is equal to 120°. Find the total surface of the cone.

Solution. We have $S_{\text{complete cone}} = S_{\text{lat}} + S_{\text{base}} = Q + \pi r^2$, where r is the radius of the base circle. Since the length of the circumference of the base circle is equal to the arc of the development, we get $2\pi r = \frac{120^\circ \cdot l \cdot 2\pi}{360^\circ}$ and $Q = \pi r l$, whence we find $l = 3r$, $Q = 3\pi r^2$ and, hence, $\pi r^2 = Q/3$ and $S = 4Q/3$.

Problem 4. A semicircle divided into three equal parts rotates about its diameter (Fig. 217). Prove that the area of the surface generated by the middle arc is equal to the sum of the areas of the surfaces described by the lateral arcs.

Solution. We join the end-points of the arcs (points B and C) to the centre of the ball and draw $[BB_1]$ perpendicular to $[AD]$ and $[CC_1]$ perpendicular to $[AD]$ (Fig. 217). When rotating, the arcs AB and CD generate surfaces of spherical segments of one base, while the arc BC generates a surface of a spherical segment of two bases.

Since $S_{\smile AB} = 2\pi R \cdot |AB_1|$, $S_{\smile CD} = 2\pi R \cdot |C_1D|$, $S_{\smile BC} = 2\pi R \times |B_1C_1|$, we have to show that $|AB_1| + |C_1D| = |B_1C_1|$.

The latter follows from the equality of central angles subtended by equal arcs: $\widehat{AOB} = \widehat{BOC} = \widehat{COD} = 60^\circ$. Consequently, $|OB_1| = |OC_1| = 0.5\,|OB|$ or $|OB_1| = 0.5\,|OA|$. Therefore $|AB_1| + |C_1D| = |B_1C_1|$.

Problem 5. A plane figure is bounded by an arc of a circle of radius R, a tangent line drawn to the circle at the end-point of this arc, and the external portion of a secant passing through the centre of the circle (Fig. 218). This figure rotates about a straight line parallel to the secant and passing through the centre of the circle. Find the volume

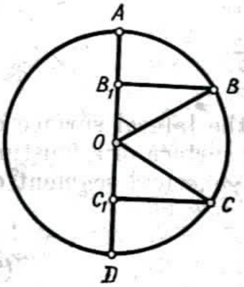


Fig. 217

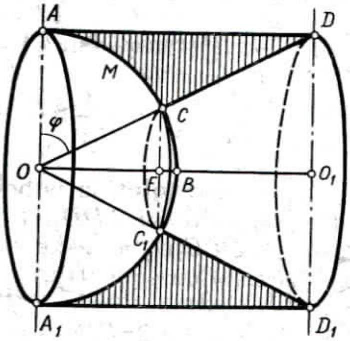


Fig. 218

and the area of the surface of the solid of revolution thus obtained if the angle between the secant and the radius is equal to φ ($\varphi > 45°$).

Solution. The solid of revolution is a circular cylinder AA_1D_1D (in which $[AA_1]$ is the diameter of the base circle and $[AD]$ is the altitude), from which there removed a solid formed by a curvilinear trapezium bounded by the curve

$$f(x)=\begin{cases}\sqrt{R^2-x^2} & 0\leqslant x\leqslant |OE|=h,\\ x\cot\varphi, & h\leqslant x\leqslant |OO_1|=H.\end{cases}$$

Using the formula computing the volume of cylinders and formula (4) of Sec. 69, we find

$$V=\pi R^2H-\pi\left\{\int_0^h (R^2-x^2)\,dx+\int_h^H x^2\cot^2\varphi\,dx\right\}=$$

$$=\pi R^2H-\pi\left\{\left[R^2x-\frac{x^3}{3}\right]_0^h+\left[\frac{x^3}{3}\cot^2\varphi\right]_h^H\right\}=$$

$$=\pi R^2H-\pi\left\{R^2h-\frac{h^3}{3}+\frac{H^3}{3}\cot^2\varphi-\frac{h^3}{3}\cot^2\varphi\right\}=$$

$$=\pi R^2H-\pi\left\{R^2h-\frac{h^3}{3\sin^2\varphi}+\frac{H^3}{3}\cot^2\varphi\right\}.$$

Since $H = |OO_1| = R\cdot\tan\varphi$, $h = R\sin\varphi$, we get

$$V = \pi R^3\left\{\tan\varphi - \sin\varphi + \frac{1}{3}\sin\varphi - \frac{1}{3}\tan\varphi\right\} =$$

$$= \frac{2}{3}\pi R^3\{\tan\varphi - \sin\varphi\} = \frac{2}{3}\pi R^3\tan\varphi\,(1-\cos\varphi) =$$

$$= \frac{4}{3}\pi R^3\tan\varphi\sin^2\frac{\varphi}{2}.$$

The solid of revolution is bounded by the lateral surface of a circular cylinder ADD_1A_1, the lateral surface of a frustum of a cone DCC_1D_1 and the surface of a spherical segment of two bases. Consequently,

$$S = 2\pi RH + \pi(R + r)\,l + 2\pi Rh, \tag{1}$$

where $r = |CE| = R\cos\varphi$, $l = |CD| = |OD| - |OC| = = \frac{R}{\cos\varphi} - R = \frac{R(1-\cos\varphi)}{\cos\varphi}$. Substituting these values in formula (1), we get

$$S = 2\pi R\cdot R\tan\varphi + \pi(R + R\cos\varphi)\cdot\frac{R(1-\cos\varphi)}{\cos\varphi} +$$

$$+ 2\pi R^2\sin\varphi = \pi R^2\tan\varphi\left[2 + \frac{\sin^2\varphi}{\cos\varphi\cdot\tan\varphi} + \frac{2\sin\varphi}{\tan\varphi}\right] =$$

$$= \pi R^2\tan\varphi\,(2 + \sin\varphi + 2\cos\varphi).$$

PROBLEMS TO CHAPTER 10

1. Find the total surface of a cube if its diagonal is equal to 27 cm.

2. Find (to within 1 cm^2) the area of the diagonal section of a cube if the area of its surface is equal to 72 dm^2.

3. The area of the complete surface of a rectangular parallelepiped whose base is a square is equal to 264 cm^2. Find the side of the base of the parallelepiped if its altitude is equal to 8 cm.

4. In a rectangular parallelepiped the ratio of the sides is 7 : 24, and the area of the diagonal section which is perpendicular to the base is equal to 50 dm^2. Find the area of the lateral surface of the parallelepiped.

5. The area of the lateral surface of a regular triangular prism is equal to 4,320 dm^2; the diagonal of the lateral face is 82 dm long. Find the altitude of the prism.

6. The base of a right parallelepiped is a rhombus. The areas of the diagonal sections perpendicular to the bases are equal to M and N. Find the area of the lateral surface of the parallelepiped.

7. The diagonal of a regular quadrangular prism is equal to 9 cm, and the area of its surface is equal to 144 cm^2. Find the side of the base and the lateral edge.

8. The area of the largest diagonal section of a regular hexagonal prism which is perpendicular to the base is equal to 1 m^2. Find the area of the lateral surface of the prism.

9. Given an oblique triangular prism whose lateral edge is equal to 80 cm. The area of the perpendicular section is equal to 384 cm^2, and the sides of the section are to one another as $4 : 13 : 15$. Find the area of the lateral surface of the prism.

10. In an oblique triangular prism the distances between the lateral edges are equal to 37 cm, 15 cm, and 26 cm, and the lateral surface is equiareal to the perpendicular section. Find the lateral edge.

11. The base of an oblique prism is a regular triangle with side a; the length of the lateral edge is equal to b. One of the lateral edges forms an angle of 45° with each of the adjacent sides of the base. Find the area of the lateral surface of the prism.

12. Given the side a of the base, determine the area of the lateral surface of a regular quadrangular pyramid whose diagonal section is equiareal to the base.

13. The centre of the upper base of a cube and the midpoints of the sides of the lower base serve as vertices of a pyramid inscribed in this cube. Determine the area of the lateral surface of the pyramid if the edge of the cube is equal to a.

14. The base of a pyramid is a rhombus with the diagonals 6 m and 8 m long; the altitude of the pyramid passes through the point of intersection of the diagonals of the rhombus and is equal to 1 m. Find the area of the lateral surface of the pyramid.

15. The base of a pyramid is a square, and its altitude passes through one of the vertices of the base. Find the area of the lateral surface of the pyramid, if the side of the base is equal to 24 dm.

16. The base of a pyramid $SABC$ is a right-angled triangle ABC. The edge AS is perpendicular to the plane of the base and is equal to 18 cm. Find (to within 1 cm^2) the area of the complete surface of the pyramid if the hypotenuse of the base $|AB|$ is equal to 25 cm and the leg $|BC|$ is equal to 7 cm.

17. The base of a pyramid is a regular hexagon with side a; one of the lateral edges is perpendicular to the plane containing the base and is equal to a. Find the area of the lateral surface of the pyramid.

18. The base of a pyramid is an equilateral triangle with side a; one of its lateral faces is also an equilateral triangle and is perpendicular to the plane of the base. Find the area of the lateral surface of the pyramid.

19. Find the area of the surface of a frustum of a regular pyramid the sides of whose bases are 18 cm and 8 cm long, and the altitude is equal to 12 cm.

20. Find the area of the surface of a frustum of a regular pyramid the sides of whose bases are 8 dm and 6 dm long, and the lateral face is inclined to the plane of the base at an angle of 60°.

21. In a frustum of a regular triangular pyramid the dihedral angle at the base is equal to 60°, the side of this base is a and the area of the surface is S. Find the side of the other base.

22. In a frustum of a regular quadrangular pyramid the areas of the bases are equal to Q and q, and the area of the lateral surface to P. Find the area of the diagonal section perpendicular to the bases.

23. The bases of a frustum of a pyramid are regular triangles with sides a and b; one of the lateral edges is perpendicular to the planes containing the bases and is equal to c. Find the area of the lateral surface of the frustum ($a = 5,\ b = 3,\ c = 1$).

24. In a frustum of a pyramid the like sides of the bases are to each other as $3:11$. What is the ratio in which the area of its lateral surface is divided by the midsection?

25. The altitude of a circular cylinder is 10 cm longer than the radius of the base, and the area of the complete surface is equal to 144 cm². Find the radius of the base and the altitude.

26. Find the relation between the altitude H of a circular cylinder and the radius R of its base circle if their sum is equal to the radius of a circle equiareal to the complete surface of this cylinder.

27. A regular hexagonal prism is inscribed in a circular cylinder. Find the ratio of the areas of their lateral surfaces.

28. Find the area of the complete surface of a cylinder circumscribed about a cube with edge a.

29. Find the area of the complete surface of an equilateral cylinder (in which the altitude is equal to the diameter of the base circle) if the area of its lateral surface is equal to 80 cm².

30. In a circular cylinder with altitude H and radius R of the base a plane is drawn parallel to the bases. Find the location of this plane at which the area of the section is the mean proportional of the areas of the parts into which the lateral surface of the cylinder is divided by this plane. Under what condition does the problem have a solution?

31. What is the mass of a cylindrical funnel whose length is equal to 15 m and the diameter to 40 cm if the funnel is made from sheet-iron, the mass of one square meter of which is equal to 7 kg? When computing the mass, 8% of the area of the surface of the funnel should be added for the seam (the result must be obtained to within 0.1 kg).

32. Given the altitude h and the area of the surface of a cone πm^2, find the radius of its base.

33. The altitude of a cone is equal to 2 dm; the angle at the vertex of its axial section is equal to 120°. Find the area of the lateral surface of the cone.

34. The greatest angle between the elements of a cone is equal to 60°. Find the ratio of the area of the lateral surface of the cone to the area of its base.

35. Find the ratio of the area of the lateral surface of an equilateral cone (its axial section is an equilateral triangle) to the area of the lateral surface of a cylinder with the same altitude.

36. Find the ratio of the generatrix to the radius of the base circle of a cone in which the lateral surface is the mean proportional between the area of the base and the complete surface of the cone.

37. The altitude of a cone is equal to 24 dm, and the radius of the base to 7 dm. Find the angle of the development of the lateral surface of this cone.

38. The ratio of the area of the complete surface of a cone to the area of its lateral surface is equal to 3 : 2. Find the greatest angle between its elements.

39. The radius of a circular sector is equal to 3 m, its angle to 120°. A conical surface is made up from this sector. Find the radius of the base of the cone thus obtained.

40. The hypotenuse and the legs of a right-angled triangle serve as diameters of three balls. Find the relation between the areas of their surfaces.

41. In a ball two parallel planes are drawn on one side of its centre. The areas of the sections are equal to 49 dm^2 and 4 m^2, the distance between them being equal to 9 dm. Find the area of the surface of the ball.

42. Prove that the area of the complete surface of a circular cone whose axial section is an equilateral triangle is equal to the area of the surface of a ball whose diameter is equal to the altitude of the cone.

43. Prove that the surface of a solid generated by rotating a square about its side is equiareal to the surface of a ball whose radius is equal to the side of the square.

44. In a frustum of a circular cone the diagonals of the axial section are mutually perpendicular, and the generatrix is inclined to the lower base at an angle α and is equal to l. Find the area of the complete surface of the frustum.

45. A ball is inscribed in a cone whose altitude is h. The line of tangency divides the surface of the ball into parts whose surfaces are to each other as $m : n$. Find the radius of the ball and the angle between the generatrix of the cone and its axis.

46. The generatrix of a cone is inclined at an angle α to the plane containing the base. A regular quadrangular pyramid is inscribed in the cone so that its vertex coincides with the vertex of the cone, and the base lies in the plane containing the base of the cone. The volume of the pyramid is equal to V. Find the area of the complete surface of the cone.

47. The area of the surface of a cone is twice the area of the surface of a ball inscribed in it. Find the angle between the generatrix of the cone and the plane containing its base.

48. Find the area of the surface of the solid obtained by rotating an isosceles triangle ABC about the line drawn from the vertex A perpendicular to the side $[AC]$ if the area Q of the triangle and the angle β between the sides $[AB]$ and $[BC]$ are known.

49. In a right circular cone a ball is inscribed, the area of whose great circle is equal to S. Find the area of the complete surface and the volume of this cone if the generatrix is inclined to the plane of its base at an angle α.

ANSWERS

To Chapter 1

1. 45°16′; 79°40′; 55°4′. **2.** 16°26′; 30°24′; 133°10′. **3.** No solutions. **4.** 14; 53°8′; 67°22′. **5.** 62.15; 21°56′; 34°4′. **6.** 1.18; 104°28′; 45°42′. **7.** 56.42; 42.31. **8.** 186.7; 158.5. **9.** 8.46; 44.56. **10.** (1) 798.4; 89°52′; 54°58′; (2) 270.5; 19°48′; 125°2′. **11.** No solution. **12.** No solution. **13.** (1) 614.4; 133°13′; 25°17′, (2) 55.6; 3°47′; 154°43′. **14.** ≈20.77; ≈55.43; 68.01. **15.** 14.56; 11.82; 5.013. **16.** 290.4; 198.6; 290.4. **17.** 34; 11; 29. **18.** 17.1; 28.7; 39.8. *Hint*: use the formula $c=r\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)$. **19.** 8.62; 5.57; 4.59; 35°39′. **20.** 15.96; 100°25′; 50°12′; 29°23′. **21.** $b=2R\sin B$. **22.** $\sin A=\frac{a}{q}$. **23.** $\sin C=\frac{\sin A}{p}$. **24.** $b=\frac{l_a\sin\left(B+\frac{A}{2}\right)}{\sin C}$. **25.** $\cos\frac{C}{2}=\frac{l_c(a+b)}{2ab}$.

26. $\frac{a\sin\frac{\beta}{2}}{\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)}$; $\frac{a\sin\frac{\alpha}{2}}{\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)}$. **27.** $\cos x=\frac{9\sqrt{2}}{20}$.

28. $\sin\widehat{ACE}=\sin 36^\circ\sqrt{\frac{2}{5+3\cos 72^\circ}}$. **29.** $\frac{c}{2\cos(\beta/2)}$; $c\sin\frac{\beta}{2}\times\tan\left(45^\circ-\frac{\beta}{4}\right)$. **30.** $\frac{c}{2\cos\alpha}$; $\frac{a}{1+\cot\frac{\alpha}{2}}$. **31.** 5; 2.

32. $\cos\alpha=\frac{4}{5}$. **33.** $\frac{d\sin\frac{3\alpha}{2}}{\sin\alpha}$; $\frac{d}{2\cos\frac{\alpha}{2}}$; $\frac{d}{2\cos\frac{\alpha}{2}}$.

34. $2[\sin A+\sin B+\sin(A+B)]$. **35.** 14; 12. **36.** 22; 14. **37.** ≈186.2 cm.

38. $\frac{c\sin 3\alpha}{\sin 2\alpha}$; $\sqrt{b^2+\frac{a^2}{4\cos^2\alpha}+\frac{ab\cos 3\alpha}{\cos\alpha}}$. **39.** 143°8′; 149°29′.

40. 126°52′; 96°44′; 136°24′. **41.** 92.5 N; 21°6′. **42.** $\frac{P}{2\sin\alpha}$; $\frac{P}{2}\cot\alpha$. **43.** ≈21 m/s. **44.** $\frac{Fl\sin\alpha}{Q\sin\beta}$.

To Chapter 3

5. 20 cm. **7.** $a/2$; $b/2$. **8.** $a\sqrt{3}/16$; $3b/4$; $b\sqrt{3}/16$. **9.** $a\sqrt{10}/3$.

To Chapter 4

2. 18. **3.** 20. **4.** 13. **5.** $|\vec{a}|=|\vec{b}|$. **6.** $\cos\alpha=1/\sqrt{3}$; $a\sqrt{3}$. **9.** $\vec{f}-\vec{c}$; $\vec{d}-\vec{f}+\vec{c}$; $\vec{f}-\vec{d}$. **10.** $\overrightarrow{AB}=-\frac{2}{3}\vec{m}-\frac{2}{3}\vec{n}$; $\overrightarrow{AC}-\frac{2}{3}\vec{n}+\frac{4}{3}\vec{m}$, $\overrightarrow{BC}=\frac{4}{3}\vec{n}+\frac{2}{3}\vec{m}$. **12.** 1 : 3. **13.** 6; 133; 51.

To Chapter 5

1. 35 cm. **2.** 82 cm. **3.** 79 dm. **4.** $4\sqrt{13}$ cm; 17 cm; 15 cm. **5.** $10\sqrt{2}\approx 14.1$ cm. **6.** 24 cm. **7.** 9 cm. **8.** 29.6 dm. **9.** 12 cm. 20 cm. **10.** a. **11.** 24 cm. **12.** 70 cm. **13.** $4\sqrt{3}/3$ cm. **14.** 1 m. **15.** 20 cm. **16.** 25 cm. **17.** 20 cm. **18.** 8 dm; 12 dm. **19.** 6 dm. **20.** 120 cm. **21.** 45°. **22.** 25 cm. **23.** $\sqrt{p^2+q^2-l^2}$. **24.** 4 dm. **25.** 20 cm. **26.** 21.2 cm. **27.** 6 dm. **28.** 24 cm. **29.** 29 cm. **30.** 41.6 cm. **31.** 10 cm. **32.** 60°. **33.** 20 cm. **34.** 10 cm. **35.** (1) 3; (2) 3, 4, 5; (3) 3, 4, 5, 6, 7; (4) 3, 4, 5, 6, 7, 8, 9, 10, 11. **36.** 73.5 cm. **37.** $2\sqrt{2}$ dm. **38.** $OA=\sqrt{(a^2+b^2+c^2)/2}$. **39.** 75 cm. **42.** 1088.4 cm^2.

To Chapter 6

1. $\overrightarrow{AB}=\{-4, 3, -1\}$, $\overrightarrow{BA}=\{4, -3, 1\}$. **2.** $|\vec{a}|=7$. **3.** $(-1, 2, 3)$. **4.** $(4, 1, 1)$. **5.** $(-1, 2, 3)$. **6.** $\overrightarrow{AM}=\{3, 4, -3\}$; $\overrightarrow{BN}=\{0, -5, 3\}$; $\overrightarrow{CP}=\{-3, 1, 0\}$. **7.** (1) 22; (2) 6; (3) -200. **8.** 31. **10.** $\alpha=-6$. **11.** $\cos\varphi=5/21$. **12.** 45°. **13.** $\arccos(-4/21)$. **14.** $|AB|=|AC|$; $\cos\hat{A}=-12/49$. **15.** $x-2y+3z+3=0$. **16.** $2x-y-z-6=0$. **17.** $x-y-3z+2=0$. **18.** $2x-y-5=0$. **19.** $z-3=0$. **20.** 90°.

To Chapter 7

1. $\sqrt{2}$ m. **3.** $a\sqrt{2}/3$. **4.** $a(\sqrt{6}-2)$. **5.** $a\sqrt{6}/3$. **6.** 50.91 dm^2. **7.** 120 cm^2; 48.5 cm^2, $24\sqrt{13}$ cm^2. **8.** (1) 6 dm; (2) 13 m; (3) 140 cm. **9.** 16 dm; 20 dm. **10.** 5 cm and 7 cm. **11.** $a\sqrt{2}$ and $2a$. **12.** 5 cm and 7 cm. **13.** $a\sqrt{6}/3$. **15.** 1872 cm^2. **16.** Q/a. **17.** 2m^2 and 3 m^2. **19.** 60 cm. **20.** $a^2\sqrt{3}/4$. **22.** $2a$ and $a\sqrt{5}$; $a^2\sqrt{3}$ and $2a^2$. **23.** $3a^2$. **24.** $Q\sqrt{2}$. **25.** 144 cm^2. **26.** 4.5 cm. **27.** 7.5 cm. **28.** 53°49′. **29.** 12 cm. **30.** $a^2/(4\pi)$. **31.** $40\sqrt{3}$ cm$^2 \approx 69.3$ cm^2. **32.** 90°. **33.** 3 dm. **34.** 1 m and 3 m. **35.** (1) $\frac{1}{2}\sqrt{4h^2+\frac{a^2}{3}}$; (2) $\sqrt{4h^2+a^2}$; $\frac{1}{2}\sqrt{4h^2+3a^2}$. **36.** (1) $\sqrt{b^2-\frac{a^2}{3}}$; (2) $\sqrt{b^2-\frac{a^2}{2}}$; (3) $\sqrt{b^2-a^2}$. **37.** 9 cm. **38.** 5 cm and 6 cm. **39.** 20 dm. **40.** 3 cm **41.** $\frac{ah}{a+h}$. **42.** 108 cm^2. **43.** 9 dm^2 **44.** 6 cm. **46.** ah; $\frac{1}{4}a\sqrt{12h^2+3a^2}$. **47.** 35 cm. **48.** 3/7 cm^2. **49.** 9 cm. **50.** 1 dm. **51.** 56 cm and 24 cm. **52.** 6 cm. **53.** 2 cm and 10 cm. **54.** $1\frac{8}{9}$ cm; $6\frac{2}{9}$ cm; $5\frac{1}{9}$ cm. **55.** 2 cm. **56.** 12 cm. **57.** $0.5(Q-q)$. **58.** 24 m^2; 80°. **59.** 39 m and 51 m. **60.** 32 m^2. **61.** 5 dm^2. **62.** $2H^2$. **63.** $R^2\sqrt{3}/2$. **64.** 1 : 2. **65.** 4 cm.

To Chapter 8

1. 16π m^2. **2.** $\pi R^2/4$. **3.** $\approx$785 km. **4.** 12 cm. **5.** 24π m. **6.** 2 cm. **8.** $\pi R\sqrt{3}$. **9.** 4π cm. **10.** 8 cm. **12.** 25 dm. **13.** $2a/3$. **14.** $\frac{a(1+3\cos^2\alpha)}{2\sin\alpha}$. **15.** $\frac{h}{\sin\alpha}$; $\frac{h}{\cos\alpha}$. **16.** $\frac{a}{2}\sqrt{\tan\left(45°-\frac{\alpha}{2}\right)}$.

To Chapter 9

1. By 30 301 cm^2. **2.** 6 cm. **3.** 3 cm. **4.** 6 m^3. **5.** $\sqrt{2}\,l^3/8$. **6.** 360 cm^3. **7.** 60 cm^3. **8.** 525 cm^3. **9.** 17 280 cm^3. **10.** 3 m^3. **11.** 6 m^3. **12.** 3.4 m; 3.4 m; 3.2 m. **13.** 12 cm^3. **14.** 7320 cm^3. **15.** 200 dm^3. **16.** $a^3/2$. **17.** $a^3\sqrt{2}/8$. **18.** am^2. **19.** 4 : 1. **20.** 1 : 2. **21.** 2 : 1. **22.** $3\pi a^3/4$. **23.** 1 : 23. **24.** 225 cm^3. **25.** 360 m^3. **26.** 400 cm^3; 180 cm^2. **27.** $3V/(4R)$. **28.** $R\sqrt[3]{4}/2 \approx 8.8R$. **29.** 0.3456 m. **30.** 3 : 4. **31.** 19.5 m^3. **32.** 128 m^2 and 50 m^2. **33.** abh. **34.** 37 m^3 and 152 m^3. **35.** 7 cm. **36.** 14 cm. **37.** 584π cm^3. **38.** $4.5\pi a^3$. **39.** $V(3+2\sqrt{2})/\pi$. **40.** $V_1=V_2=\pi R^2H/36$; $V_3=V_4=5\pi R^2H/36$. **41.** $32\pi a^3/81$. **42.** 4847 pcs. **43.** 3 : 2 : 1. **44.** $R\sqrt[3]{15}\approx 2.5R$. **45.** 45π cm^3 and 243π cm^3. **46.** 5 : 16. **47.** 288 π cm^3. **48.** $R/2$. **49.** $R\sqrt{3}/3$. **50.** 112.5π dm^3. **51.** $38\pi/3\approx 40$ m^3 or $434\pi/3\approx 454$ m^3. **52.** $V_{\text{cone}} : V_{\text{sphere}} = 2\sin^2\alpha\cos^4\alpha : 1$. **53.** $\pi r^3\sin\alpha/(12\cos^5\alpha\cos 2\alpha)$. **54.** $\arcsin(2/\sqrt{7})$. **55.** $\arccos\sqrt[4]{1/2}$.

To Chapter 10

1. 1458 cm^2. **2.** 1697 cm^2. **3.** 6 cm. **4.** 124 dm^2. **5.** 18 dm or 80 dm. **6.** $2\sqrt{M^2+N^2}$. **7.** 6 cm and 3 cm or 4 cm and 7 cm. **8.** 3 m^2. **9.** 102.4 dm^2. **10.** 2 cm. **11.** $ab(\sqrt{2}+1)$. **12.** $3a^2$. **13.** $3a^2/4$. **14.** 26 m^2. **15.** 343 dm^2. **16.** 630 cm^2. **17.** $a^2(6+\sqrt{7})/2 \approx 4.3\,a^2$. **18.** $a^2(\sqrt{3}+\sqrt{5})/4 \approx 1.40a^2$. **19.** 1064 cm^2. **20.** 156 dm^2. **21.** $\sqrt{3a^2-\frac{4S}{\sqrt{3}}}$. **22.** $\sqrt{2[P^2-(Q-q)^2]}/4$. **23.** $0.25(a+b)[4c+\sqrt{4c^2+3(a-b)^2}]=16$. **24.** $5:9$. **25.** 3 cm and 14 cm. **26.** $H=R$. **27.** $\pi:3$. **28.** $\pi a^2(\sqrt{2}+1)$. **29.** 120 cm^2. **30.** $0.5(H \pm \sqrt{H^2-R^2})$, $H \geqslant R$. **31.** 142.5 kg. **32.** $m^2/\sqrt{2m^2+h^2}$. **33.** 43.5π dm^2. **34.** $2:1$. **35.** $2:3$. **36.** $(1+\sqrt{5}):2$. **37.** 100°48′. **38.** 60°. **39.** 1 m. **40.** The greater surface area is equal to the sum of two others. **41.** 25π m^2. **44.** $2\pi l^2 \sin\left(15^\circ+\frac{\alpha}{2}\right) \times \cos\left(15^\circ-\frac{\alpha}{2}\right)$. **45.** $\alpha=\arcsin\frac{n-m}{n+m}$. **46.** $\frac{\pi}{2}\cot\frac{\alpha}{2}\sqrt[3]{18V^2\tan\alpha}$. **47.** $2\arctan\frac{\sqrt{2}}{2}$. **48.** $4\pi Q\cot\left(45^\circ-\frac{\beta}{4}\right)$. **49.** $\frac{S\cot^3\frac{\alpha}{2}}{\cot\alpha}$; $\frac{S\sqrt{S}\cot^3\frac{\alpha}{2}}{3\sqrt{\pi}\cot\alpha}$.

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TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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